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A CLASS OF SEMI-INNER PRODUCTS AND APPLICATIONS (I)

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Abstract. Some theorems of decomposition for a class of smooth normed linear spaces endowed with a derivable semi-inner product are given.

0. Introduction. Let $(X, \|\cdot\|)$ be a real normed linear space and $f : X \to R$ the function given by $f(x) := 1/2\,\|x\|^2$ for $x \in X$. We recal the notion of semi-inner product in the sense of Tapia (see [1] pp. 389–390 or [7]):

0.1. DEFINITION. The mapping $(,)_T : X \times X \to \mathbb{R}$ given by

$$(x, y)_T := \lim_{t \downarrow 0} [f(y + tx) - f(y)]/t, \quad x, y \in X; \tag{0.1}$$

is called semi-inner product in the sense of Tapia or T-semi-inner product, for short.

We list some usual properties of T-semi-inner products.

0.2. PROPOSITION. *If $(X, \|\cdot\|)$ is a real normed linear space, then*

(i) $(x, x)_T = \|x\|^2$ *for* $x \in X$;

(ii) $(\alpha x, \beta y)_T = \alpha\beta(x, y)_T$ for $\alpha, \beta \in \mathbb{R}$, $\alpha\beta \geq 0$ *and* $x, y \in X$;

(iii) $(\alpha x + y, x)_T = \alpha\|x\|^2 + (y, x)_T$ *for* $\alpha \in \mathbb{R}$ *and* $x, y \in X$;

(iv) $(-x, y)_T = (x, -y)_T$ *if* $x, y \in X$;

(v) $(x + y, z)_T \leq \|x\|\,\|z\| + (y, z)_T$ *for all* $x, y, z \in X$;

(vi) $|(x, y)_T| \leq \|x\|\,\|y\|$ *if* $x, y \in X$;

(vii) $(,)_T$ is subadditive and continuous in the first argument.

For the proof of the previous properties of T-semi-inner products we send to [6] pp. 38–39 and [1] p. 390.

The following characterization of smooth normed linear spaces in terms of T-semi-inner products holds.

0.3. PROPOSITION. ([1] p. 392) *Let* $(X, \|\cdot\|)$ *be a real normed linear space. Then the following sentences are equivalent*:

(i) *The norm is Gâteaux differentiable on* $X \setminus \{0\}$, *i.e.*, $(X, \|\cdot\|)$ *is a smooth normed linear space*;

(ii) $(,)_T$ *is homogeneous in the second argument*;

(iii) $(,)_T$ *is homogeneous in the first argument*;

(iv) $(,)_T$ *is linear in the first variable*;

If $(X, \|\cdot\|$ *is a smooth normed linear space, then the following identity holds*:

$$(0.2)\qquad (y, x)_T = \lim_{t \to 0} [(x, x + ty)_T - \|x\|^2]/t \text{ for all } x, y \in X.$$

For the proof of this fact see Lemma 1.2 of [3].

0.4. DEFINITION ([1] p. 386, [5]) Let X be a real linear space. A mapping $(,)_L : X \times X \to \mathbb{R}$ is called semi-inner product in the sense of Lummer (L-semi-inner product) if the following conditions are satisfied:

(i) $(x + y, z)_L = (x, z)_L + (y, z)_L$ for $x, y, z \in X$;

(ii) $(\lambda x, y)_L = \lambda (x, y)_L$ for $\lambda \in \mathbb{R}$ and $x, y \in X$;

(iii) $(x, x)_L > 0$ if $x \neq 0$;

(iv) $|(x, y)_L|^2 \leqslant (x, x)_L (y, y)_L$ for $x, y \in X$;

(v) $(x, \lambda y)_L = \lambda (x, y)_L$ for $\lambda \in \mathbb{R}$ and $x, y \in X$.

We note that the mapping $X \ni x \overset{\|\cdot\|}{\to} (x, x)_L^{1/2} \in \mathbb{R}_+$ is a norm on X and the functional given by $X \ni x \overset{f_y}{\to} (x, y)_L \in \mathbb{R}$ is a continuous linear functional on the normed linear space $(X, \|\cdot\|)$.

0.5. PROPOSITION ([1] p. 387). *Let* $(X, \|\cdot\|)$ *be a normed linear space. Then* $(X, \|\cdot\|)$ *is a smooth normed linear space iff there exists a unique L-semi-inner product which generates the norm* $\|\cdot\|$.

By the use of the notion of continuous L-semi-inner product, i.e. a L-semi-inner product which generates the norm and and satisfies the assumption:

$$(0.3)\qquad \lim_{t \to 0} (y, x + ty)_L = (y, x)_L \text{ for all } x, y \in X$$

we have the following characterization of smooth normed linear spaces.

0.6. PROPOSITION. ([1] p. 387). *Let* $(X, \|\cdot\|)$ *be a real normed linear space and let* $(,)_L$ *be a L-semi-inner product which generates the norm* $\|\cdot\|$. *Then* $(,)_L$ *is continuous iff* $(X, \|\cdot\|)$ *is a smooth normed linear space.*

It is known that the semi-inner product in the Tapia sense is a L-semi-inner product iff the normed linear space is smooth (see [1] p. 392).

Now, we recall the well-known concept of orthogonality in the sense of Birkhoff.

0.7. DEFINITION. Let $(X, \|\cdot\|)$ be a normed linear space and $x, y \in X$. The element x is said to be orthogonal over y if

$$(0.4)\qquad \|x + ty\| \geqslant \|x\| \text{ for all } t \in \mathbb{R}.$$

We denote this by $x \perp_B y$.

By the theorem of R. C. James (see for example [8] p. 85):

0.8. THEOREM. *Let* $(X, \|\cdot\|)$ *be a real normed linear space and* α *a given real number. Then the following sentences are equivalent*

(i) $x \perp_B \alpha x + y$;

(ii) $-\tau(x, -y) \leqslant -\alpha\|x\| \leqslant \tau(x, y)$

where $\tau(x, y) := \lim_{t \downarrow 0} (\|x + ty\| - \|x\|)/t = 1/\|x\|(y, x)_T$ *for* $x, y \in X$ *and* $x \neq 0$;

we conclude that in smooth normed linear spaces Birkhoff's orthogonality is equivalent with Tapia's orthogonality and with Lumer's orthogonality respectively, i.e.,

$$(0.5.)\qquad x \perp_B y \text{ iff } (y, x)_T = 0 \text{ iff } (y, x)_L = 0.$$

Finally, we recall Tapia's theorem of representation (see for example [1] p. 400):

0.9. THEOREM. *Let* $(X, \|\cdot\|)$ *be a real Banach space. Then the following sentences are equivalent*:

(i) $(X, \|\cdot\|)$ *is a smooth reflexive Banach space*;

(ii) *For any* $f \in X^*$ *there exists an element* $u_f \in X$ *such that*

$$(0.6)\qquad f(x) = (x, u_f)_T \text{ for all } x \in X$$

and $\|f\| = \|u_f\|$.

1. Derivable semi-inner products. Let $(X, \|\cdot\|)$ be a normed linear space over the real number field. We give the following definition.

1.1. DEFINITION. The T-semi-inner product is said to be continuous on X if the following conditions holds

$$(1.1)\qquad \lim_{t \to 0} (y, x + ty)_T = (y, x)_T \text{ for all } x, y \in X$$

Now, we can give the following charterization of smooth normed linear spaces in terms of continuous T-semi-inner products.

1.2. PROPOSITION. *Let* $(X, \|\cdot\|)$ *be a real normed linear space. Then the following sentences are equivalent*:

(i) $(,)_T$ *is continuous on* X;

(ii) $(X, \|\cdot\|)$ *is a smooth normed linear space.*

Proof. "(i) $\Rightarrow$ (ii)". By the properties of T-semi-inner product we have

(1.2) $$(y, x)_T/\|x\| \leqslant (\|x + ty\| - \|x\|)/t \leqslant (y, x + ty)_T/\|x + ty\|$$

and

(1.3) $$(y, x + sy)_T/\|x + sy\| \leqslant (\|x + sy\| - \|x\|)/s \leqslant (y, x)_T/\|x\|$$

for all $x, y \in X$, $x \neq 0$ and $t > 0$, $s < 0$.

Then we obtain:

$$\lim_{t \downarrow 0}(\|x + ty\| - \|x\|)/t = (y, x)_T/\|x\| \text{ and}$$

$$\lim_{s \uparrow 0}(\|x + sy\| - \|x\|)/s = (x, y)_T/\|x\|$$

for all $x, y \in X$, $x \neq 0$, i.e., the norm is Gâteaux differentiable on $X \setminus \{0\}$.

"(ii) $\Rightarrow$ (i)". If $(X, \|\cdot\|)$ is a smooth normed linear space, then $(,)_T$ is the unique L-semi-inner product which generates the norm $\|\cdot\|$ (see [1] p. 392) and by Proposition 0.5. we deduce that $(,)_T$ is continuous on X.

The proposition is proven.

1.3. DEFINITION. Let $(X, \|\cdot\|)$ be a smooth linear space and $(,)$ the semi-inner product in the sense of Tapia or Lumer which generates the norm $\|\cdot\|$. Then $(,)$ is called derivable on X if the following limit:

(1.4) $$(y, x)' := \lim_{t \to 0} [(y, x + ty) - (y, x)]/t$$

exists for all x, y in X.

Now, we introduce the following class of smooth normed linear spaces.

1.4. DEFINITION. A smooth normed linear space is called of (D)-type if the semi-inner product in the sense of Tapia or Lumer is derivable.

1.5 EXAMPLES. 1. Every inner-product space $(X; (,))$ is a smooth normed linear space of (D)-type.

Indeed, since for any $x, y \in X$ we have:

(1.5) $$(y, x)' = \lim_{t \to 0} [(y, x + ty) - (y, x)]/t = \|y\|^2.$$

2. Let $(X; (,))$ be a prehilbertian space over the real number field and $A : X \to X$ be a nonlinear operator with the properties:

(a) $A(\alpha x) = \alpha A x$ for $\alpha \in \mathbb{R}$ and $x \in X$;

(aa) $(x, Ax) \geqslant 0$ for $x \in X$ and $(x, Ax) = 0$ implies $x = 0$;

(aaa) $|(x, Ay)|^2 \leqslant (x, Ax)\,(y, Ay)$ for all $x, y \in X$;

(av) $\lim_{t \to 0} A(x + ty) = Ax$ in $(X, \|\cdot\|)$ for all $x, y \in X$;

(v) the Gâteaux differential $(\nabla A)(x) \cdot y := \lim_{t \to 0} [A(x + ty) - Ax]/t$ exists for all $x, y \in X$;

then $(X, \|\cdot\|_A)$ where $\|x\|_A := (x, Ax)^{1/2}$ for $x \in X$ is a smooth normed linear space of (D)-type.

Indeed, putting $(,)_A : X \times X \to \mathbb{R}$, $(x, y)_A := (x, Ay)$, then $(,)_A$ is a continuous L-semi-inner product and since

(1.6) $$(y, x)'_A = (y, (\nabla A)(x) \cdot y) \text{ for all } x, y \in X,$$

then $(X, \|\cdot\|_A)$ is a smooth normed linear space of (D)-type.

Now, we shall give some usual properties of semi-inner product derivative in a smooth normed linear space of (D)-type.

1.6. PROPOSITION. *If $(X, \|\cdot\|)$ is as above, then:*

(i) $(y, y)' = \|y\|^2$ *for all* $y \in X$;

(ii) $(y, 0)' = \|y\|^2$ *for all* $y \in X$;

(iii) $(\alpha y, x)' = \alpha^2 (y, x)'$ *for* $\alpha \in \mathbb{R}$ *and* $x, y \in X$;

(iv) $(y, \alpha x)' = (y, x)'$ *for all* $\alpha \in \mathbb{R} \setminus \{0\}$ *and* x, y *in* X;

(v) $\|x\|^2 (y, x)' \geqslant (y, x)^2$ *for all* $x, y \in X$.

Proof. The sentences (i) and (ii) are obvious by the definition of the semi-inner product derivative.

(iii). If $\alpha = 0$ the identity is valid. Now, let us suppose $\alpha \neq 0$. Then

$$(\alpha y, x)' = \lim_{t \to 0} [(\alpha y, x + \alpha t y) - (\alpha y, x)]/t = \alpha \lim_{t \to 0} [(y, x + \alpha t y) - (y, x)]/t = \alpha^2 \lim_{t \to 0} [(y, x + \alpha t y) - (y, x)]/\alpha t = \alpha^2 \lim_{s \to 0} [(y, x + sy) - (y, x)]/s = (y, x)' \text{ for all } x, y \in X.$$

(iv). If $\alpha \neq 0$, then we have:

$$(y, \alpha x)' = \lim_{t \to 0} [(y, \alpha x + ty) - (y, \alpha x)]/t = \alpha \lim_{t \to 0} [(y, x + t/\alpha y) - (y, x)]/t = \lim_{t \to 0} [(y, x + t/\alpha y) - (y, x)]/(t/\alpha) = \lim_{s \to 0} [(y, x + sy) - (y, x)]/s = (y, x)' \text{ for all } x, y \in X.$$

(v). From relation (1.2) we have

$$(y, x + ty) - (y, x) \geqslant (y, x)\,(\|x + ty\| - \|x\|)/\|x\|, \quad x, y \in X, \quad x \neq 0$$

from where there results

$$[(y, x + ty) - (y, x)]/t \geqslant (y, x)\,(\|x + ty\| - \|x\|)/(t\|x\|) \text{ for } x, y \in X,$$

$x \neq 0$ and $t > 0$. Then we obtain

$$(y, x)' \geqslant (y, x)^2/\|x\|^2 \text{ for } x, y \in X \text{ and } x \neq 0$$

and the proposition is proven.

Another result is embodied in the next proposition.

1.7. PROPOSITION. *Let $(X, \|\cdot\|)$ be a smooth normed linear space of (D)-type and x, y two given elements in X. Then the mapping*

$$(1.7)\qquad \varphi_{x,y} : \mathbb{R} \to \mathbb{R}_+,\quad \varphi_{x,y}(t) := \|x + ty\|^2,\ t \in \mathbb{R},$$

is derivable of two orders on $\mathbb{R}$ and the second derivative is nonnegative on $\mathbb{R}$. In addition,

(i) $$\varphi'_{x,y}(t) = 2(y, x + ty) \text{ for all } t \in \mathbb{R};$$

and

(ii) $$\varphi''_{x,y}(t) = 2(y, x + ty)' \text{ for all } t \in \mathbb{R}.$$

Proof. Let $t_0 \in \mathbb{R}$. Then we have

$$\lim_{t \to t_0} [\varphi_{x,y}(t) - \varphi_{x,y}(t_0)]/(t - t_0) = \lim_{h \to 0} (\|x + t_0 y + hy\|^2 - \|x + t_0 y\|^2)/h = 2(y, x + t_0 y) \text{ for all } x, y \in X.$$

The proof of relation (ii) is similar and we omit the details.

The second derivative of $\varphi_{x,y}$ is nonnegative from the property (v) of Proposition 1.6.

In the sequel, we shall give a simple proof for the equivalence between Birkhoff's orthogonality and the orthogonality in the sense of semi-inner product for the smooth normed linear spaces of (D)-type.

1.8. PROPOSITION. *Let $(X, \|\cdot\|)$ be a smooth space of (D)-type. Then the following assertions are equivalent:*

(i) $$x \perp_B y;$$

(ii) $$x \perp y \text{ i.e. } (y, x) = 0.$$

Proof. By Taylor's theorem, we have:

$$\|x + ty\|^2 = \|x\|^2 + 2(y, x)t + (y, x + \xi_t y)' t^2$$

where ξ_t is between 0 and t.

If $x \perp_B y$ then $\|x + ty\|^2 \geq \|x\|^2$ for all $t \in \mathbb{R}$ from where there results

$$t^2(y, x + \xi_t y)' + 2(y, x)t \geq 0 \text{ for all } t \in \mathbb{R}$$

which implies $(y, x) = 0$, i.e., $x \perp y$.

If $x \perp y$, then

$$\|x + ty\|^2 - \|x\|^2 = (y, x + \xi_t y)' t^2 \geq 0 \text{ for all } t \in \mathbb{R},$$

i.e., $x \perp_B y$ and the proposition is proven.

Further on, we shall give a characterization theorem of prehilbertian spaces in the class of smooth normed linear spaces of (D)-type.

1.9. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth space of (D)-type. Then the following sentences are equivalent:*

(i) *$(X, \|\cdot\|)$ is prehilbertian space;*

(ii) *The mapping $\psi_{x,y} : \mathbb{R} \to \mathbb{R}$, $\psi_{x,y}(t) = (y, tx)'$ is continuous in 0 for all $x, y \in X$;*

(iii) *For every $x, y \in X$ there exists a sequence $\alpha_n \in \mathbb{R} \setminus \{0\}$, $\alpha_n \to 0$ such that $\lim_{n\to\infty} (y, \alpha_n x)' = (y, 0)'$;*

(iv) *For every $x, y \in X$ we have $(y, x)' = \|y\|^2$.*

Proof. "(i) ⇒ (ii)". It is obvious observing that $(y, x)' = \|y\|^2$.
"(i) ⇒ (iii)". It is obvious.
"(iii) ⇒ (iv)". Let $x, y \in X$ and $\alpha_n \in \mathbb{R} \setminus \{0\}$ with the above property. Since $(y, x)' = (y, \alpha_n x)'$, we obtain:

$$(y, x)' = \lim_{n\to\infty} (y, \alpha_n x)' = (y, 0)' = \|y\|^2.$$

"(iv) ⇒ (i)". By Taylor's theorem, we have:

$$\|x + ty\|^2 = \|x\|^2 + 2(y, x)t + \|y\|^2 t^2 \text{ for all } t \in \mathbb{R},$$

which implies

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for all } x, y \in X$$

and then $(X, \|\cdot\|)$ is prehilbertian.

The theorem is proven.

2. Semi-inner products with bounded derivative.

Let $(X, \|\cdot\|)$ be a smooth normed linear space of (D)-type and $(,)$ the semi-inner product which generates the norm $\|\cdot\|$.

2.1. DEFINITION. The semi-inner product has a bounded derivative on X if there exists a real number k such that $k \geq 1$ and

$$(2.1)\qquad (x, y)' \leq k^2 \|x\|^2 \text{ for all } x, y \in X.$$

The best number k such that (2.1) is valid is called the boundedness module of the derivative $(,)'$ and we denote this number by $k_{(,)'}$.

Now, we can define the following class of smooth spaces of (D)-type.

2.2. DEFINITION. A smooth normed linear space is called of (BD)-type if the semi-inner product which generates the norm has a bounded derivatives.

2.3. EXAMPLES. 1. Every inner-product space $(X; (,))$ is a smooth normed linear space of (BD)-type.

2. Let $(X; (,))$ be a prehilbertian space over the real number field and $A : X \to X$ a nonlinear operator with the properties: (a), (aa), (aaa), (av), (v) (see Example 1.5.2) and

$$(\text{va})\qquad \|(\nabla A)(x) \cdot y\| \leq k^2 \|y\| \text{ for all } x, y \in X,$$

then $(X, \|\cdot\|_A)$ where $\|x\|_A := (x, Ax)^{1/2}$ for all $x \in X$ is a smooth normed linear space of (BD)-type.

The following result gives a characterization of inner-product spaces in the class of smooth normed linear space of (BD)-type.

2.4. THEOREM. *Let $(X, \|\cdot\|)$ be a smooth space as above. Then the following sentences are equivalent:*

(i) *$(X, \|\cdot\|)$ is an inner product spaces;*

(ii) *The boundedness module of $(,)'$ is 1.*

Proof. "(i) ⇒ (ii)". It is obviously since $(x, y)' = \|x\|^2$ for all $x,y \in X$.

"(ii) ⇒ (i)". By Taylor's formula we obtain

$$\|x + y\|^2 \leqslant \|x\|^2 + 2(y, x) + \|y\|^2 \text{ for all } x, y \in X$$

which implies

$$\|x + y\|^2 \leqslant \|x\|^2 + 2(x, y) + \|y\|^2 \text{ for all } x, y \in X.$$

Then we obtain

$$\|x + ty\|^2 \leqslant \|x\|^2 + 2(x, y)t + \|y\|^2t^2 \text{ for all } x, y \in X \text{ and } t \in \mathbb{R}.$$

Let $t \in \mathbb{R}$, $t > 0$. Then:

$$(\|x + ty\|^2 - \|x\|^2)/t \leqslant (x, y) + t\|y\|^2$$

which gives for $t \to 0$, $t > 0$:

$$(y, x) \leqslant (x, y) \text{ for all } x, y \in X,$$

and then by symmetry: $(x, y) = (y, x)$ and $(X, \|\cdot\|)$ is an inner product space.

The theorem is proven.

2.5. DEFINITION. Let $(X, \|\cdot\|)$ be a smooth normed linear space of (BD)-type and $k_0 := k_{(,)'}$ the boundedness module of semi-inner product derivative. If $\varepsilon \in [0,1)$, then the element $x \in X$ is said to be $\varepsilon - k_0$ — orthogonal over y if

$$|(y, x)| \leqslant k_0\varepsilon \|x\| \|y\|, \tag{2.2}$$

and we denote this fact by $x \underset{\varepsilon}{\overset{k_0}{\perp}} y$.

2.6. REMARK. If $(X; (,))$ is an inner product space, then in (2.2) we can find $k_0 = 1$. Then we have

$$|(y, x)| \leqslant \varepsilon\|x\| \|y\| \tag{2.3}$$

which will be denoted by $x \underset{\varepsilon}{\perp} y$.

If in the previous definitions we consider $\varepsilon = 0$, then we recapture the usual orthogonality in the sense of semi-inner product or the usual orthogonality in prehilbertian spaces, respectively.

Further on, we shall give the following generalization of Birkhoff's orthogonality.

2.6. DEFINITION. Let $(X, \|\cdot\|)$ be a normed linear space, $\varepsilon \in [0,1)$ and $x,y \in X$. The element $x \in X$ is said to be ε-Birkhoff orthogonal over the element y and we write that $x \underset{\varepsilon}{\perp}_B y$ if

$$\|x + ty\| \geqslant (1 - \varepsilon)\|x\| \text{ for all } t \in \mathbb{R}. \tag{2.4}$$

The following proposition contains a connection between $\varepsilon - k_0$-orthogonality and ε-Birkhoff orthogonality.

2.7. PROPOSITION. *Let $(X, \|\cdot\|)$ be a smooth normed linear space of (BD)-type and k_0 the boundedness module of semi-inner product derivative. If $x,y \in X$ and $\varepsilon \in [0,1)$, then the following sentences are valid:*

(i) $x \underset{\varepsilon}{\perp}_B y$ *implies* $x \underset{\delta(\varepsilon)}{\overset{k_0}{\perp}} y$ *with* $\delta(\varepsilon) := [\varepsilon(2 - \varepsilon)]^{1/2}$;

(ii) $x \underset{\eta(\varepsilon)}{\perp}_B y$ *implies* $x \underset{\varepsilon}{\overset{k_0}{\perp}} y$ *with* $\eta(\varepsilon) := 1 - (1 - \varepsilon^2)^{1/2}$.

Proof. We shall start to Taylor's expansion:

$$\|x + ty\|^2 = \|x\|^2 + 2(y, x)t + (y, x + \xi_t y)'t^2 \text{ for } t \in \mathbb{R},$$

where ξ_t is between 0 and t.

(i). If $x \underset{\varepsilon}{\perp}_B y$, then

$$(1 - \varepsilon^2)\|x\|^2 \leqslant \|x + ty\|^2 \text{ for all } t \in \mathbb{R},$$

which implies

$$(\varepsilon^2 - 2\varepsilon)\|x\|^2 \leqslant \|x + ty\|^2 - \|x\|^2 \leqslant 2(y, x)t + (y, x + \xi_t y)'t^2 \leqslant$$
$$\leqslant 2(y, x)t + k_0^2\|y\|^2t^2 \text{ for all } t \in \mathbb{R},$$

i.e.,

$$k_0^2\|y\|^2t^2 + 2(y, x)t + \varepsilon(2 - \varepsilon)\|x\|^2 \geqslant 0 \text{ for all } t \in \mathbb{R}$$

which implies

$$1/4\Delta = (y, x)^2 - k_0^2\varepsilon(2 - \varepsilon)\|x\|^2\|y\|^2 \leqslant 0$$

from where there results: $x \underset{\delta(\varepsilon)}{\overset{k_0}{\perp}} y$ with $\delta(\varepsilon) := [\varepsilon(2 - \varepsilon)]^{1/2}$

The second affirmation follows by (i) substituting ε by $\eta(\varepsilon) \in [0,1)$. We omit the details.

Further, we shall analyse the prehilbertian case.

2.8. PROPOSITION. *Let $(X; (,))$ be a real prehilbertian space and $\varepsilon \in [0,1)$. Then the following affirmations hold:*

(i) $x \underset{\varepsilon}{\perp}_B y$ *iff* $x \underset{\delta(\varepsilon)}{\perp} y$ *where* $\delta(\varepsilon) := [\varepsilon(2 - \varepsilon)]^{1/2}$;

(ii) $x \underset{\eta(\varepsilon)}{\perp} y$ *iff* $x \underset{\varepsilon}{\perp} y$ *where* $\eta(\varepsilon) := 1 - (1 - \varepsilon^2)^{1/2}$.

Proof. (i). We must only prove the implication $(\Leftarrow)$. It is clear that:

$$\|x+ty\|^2 = \|x\|^2 + 2(y,x)t + \|y\|^2t^2 \text{ for all } t \in \mathbb{R}.$$

If $x \underset{\delta(\varepsilon)}{\perp} y$, then

$$\|y\|^2t^2 + 2(y,x)t + \varepsilon(2-\varepsilon)\|x\|^2 \geq 0 \text{ for all } t \in \mathbb{R}$$

since $\Delta \leq 0$, from where there results:

$$\|x+ty\|^2 - \|x\|^2 \geq (\varepsilon^2 - 2\varepsilon)\|x\|^2 \text{ for all } t \in \mathbb{R},$$

which gives $x \underset{\varepsilon}{\perp}{}_B\, y$.

The second affirmation is obvious.

The proposition is proven.

3. **The ε-k_0-orthogonal decomposition.** We shall begin our considerations with some general results which work in the normed linear spaces.

Let $(X, \|\cdot\|)$ be a normed linear space and A its nonvoid subset. By $A^{\perp^B_\varepsilon}$ we shall denote the set given by:

$$A^{\perp^B_\varepsilon} = \{y \in X \mid y \perp_B x \text{ for all } x \in X\}, \tag{3.1}$$

where ε is a given real number in $[0,1)$. This set will be called the ε-Birkhoff orthogonal complement of A.

It is easy to see that $0 \in A^{\perp^B_\varepsilon}$ and $A \cap A^{\perp^B_\varepsilon} \subseteq \{0\}$ for all $\varepsilon \in [0,1)$.

The following lemma is valid.

3.1. Lemma. *Let $(X, \|\cdot\|)$ be a normed linear space and E be its closed linear subspace. Suppose $E \neq X$. Then for every $\varepsilon \in (0,1)$ the ε-Birkhoff orthogonal complement of E is nonzero.*

Proof. Let $\bar{y} \in X \setminus E$. Since E is closed, $d(y,E) = d > 0$. Thus there exists $\bar{y}_\varepsilon \in X$ such that $d \leq \|y - y_\varepsilon\| \leq d/(1-\varepsilon)$.

Putting $x_\varepsilon := \bar{y} - y_\varepsilon$, we have $x_\varepsilon \neq 0$, and for every $y \in E$ and $\lambda \in K$:

$$\|x_\varepsilon + \lambda y\| = \|\bar{y} - y_\varepsilon + \lambda y\| = \|\bar{y} - (y_\varepsilon - \lambda y)\| \geq d \geq (1-\varepsilon)\|x_\varepsilon\|$$

what means that $x_\varepsilon \in E^{\perp^B_\varepsilon}$ and the lemma is proven.

The following decomposition theorem in general normed linear spaces holds.

3.2. Theorem. *Let $(X, \|\cdot\|)$ be a normed linear space and E be its closed linear subspace. Then for every $\varepsilon \in (0,1)$ the following decomposition holds:*

$$X = E + E^{\perp^B_\varepsilon}. \tag{3.2}$$

Proof. Suppose $E \neq X$ and $x \in X$.

If $x \in E$, then $x = x + 0$ with $x \in E$ and $0 \in E^{\perp^B_\varepsilon}$.

If $x \notin E$, then there exists $y_\varepsilon \in E$ such that $0 < d = d(x,E) \leq \|x - y_\varepsilon\| \leq d/(1-\varepsilon)$.

Since $x_\varepsilon = x - y_\varepsilon \in E^{\perp^B_\varepsilon}$ (see the proof of the above lemma) we obtain $x = y_\varepsilon + x_\varepsilon$ and relation (3.2) is valid.

Further on, we shall apply these results in the particular case of smooth normed linear space of (BD)-type.

Let $(X, \|\cdot\|)$ be a smooth normed space as above and A a nonempty subset in X. Then by $A^{\perp^B_\varepsilon}$ we shall denote the set:

$$A^{\overset{k_0}{\underset{\varepsilon}{\perp}}} := \{y \in X \mid y \overset{k_0}{\underset{\varepsilon}{\perp}} x \text{ for all } x \in A\},\ \varepsilon \in [0,1), \tag{3.3}$$

where k_0 is the boundedness module of $(,)'$, which will be called the ε-k_0-orthogonal complement of A in X.

3.3. Lemma. *Let $(X, \|\cdot\|)$ be a smooth normed linear space of (BD)-type and E be its closed linear subspace, $\varepsilon \in (0,1)$ and $E \neq X$, then the ε-k_0-ortohgonal complement of E is nonzero.*

Proof. Let $\varepsilon \in (0,1)$ and $\eta(\varepsilon) :/1 - (1-\varepsilon^2)^{1/2}$. Then $\eta(\varepsilon) \in (0,1)$. Applying Lemma 3.1 for $\eta(\varepsilon)$, there exists an element $x_\varepsilon \neq 0$ and $x_\varepsilon \in E^{\perp^B_{\eta(\varepsilon)}}$. Since $E^{\perp^B_{\eta(\varepsilon)}} \subseteq E^{\overset{k_0}{\underset{\varepsilon}{\perp}}}$ (see Proposition 2.8), the lemma is proven.

Finally, we have the following ε-k_0-orthogonal decomposition of X.

3.4. Theorem *Let $(X, \|\cdot\|)$ be a smooth normed linear space of (BD)-type, E be its closed linear subspace in X and $\varepsilon \in (0,1)$. Then the following decomposition holds:*

$$X = E + E^{\overset{k_0}{\underset{\varepsilon}{\perp}}}. \tag{3.4}$$

Proof. Let $\varepsilon \in (0,1)$ and $\eta(\varepsilon) = 1 - (1-\varepsilon^2)^{1/2} \in (0,1)$. If $x \in X$, then there exists $x_\varepsilon \in E$ and $y_\varepsilon \in E^{\perp^B_{\eta(\varepsilon)}}$ such that $x = x_\varepsilon + y_\varepsilon$ (see Theorem 3.2). Since $E^{\perp^B_{\eta(\varepsilon)}} \subseteq E^{\overset{k_0}{\underset{\varepsilon}{\perp}}}$ (see Proposition 2.8) we obtain $x = x_\varepsilon + y_\varepsilon$ with $x_\varepsilon \in E$ and $y_\varepsilon \in E^{\overset{k_0}{\underset{\varepsilon}{\perp}}}$ and the theorem is proven.

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