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BIFURCATION MANIFOLDS IN A MULTIPARAMETRIC EIGENVALUE PROBLEM FOR LINEAR HYDROMAGNETIC STABILITY THEORY*

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Summary — Zusammenfassung. *Bifurcation surfaces in a multiparametric eigenvalue problem from linear hydromagnetic stability theory.* The neutral surface separating domains of linear stability and instability of a Couette flow under an axial magnetic field is investigated analytically and numerically. A special attention is paid to the involved bifurcation.

Verzweigungsoberflächen in eine mancheparametrische Eigenwertproblem auf die Theorie des hydromagnetische stabilität. Die neutrale Öberfläche das scheide die Gebiete von linear Stabilität und Instabiltät eines Couette Bewegung hinten einen axial magnetisch Feld is ergründet analytisch und numerisch. Eine spezielle Aufmerksamkeit ist schenken der Verzweigungsfrage.

1. **The method**. Most of the eigenvalue problems governing the linear stability of motion of continua (occurring in hydrodynamics, hydromagnetics, elasticity, aeroelasticity, vibration theory) consist of linear ordinary differential equations of high order ($n \geqslant 8$), with constant coefficients depending on several physical parameters a, Q, $T, \ldots$, and some homogeneous boundary conditions. This high order suggested the application of various methods, involving Fourier series, to solve these problems. However, in spite of the fact that n is large the simplest classical method may be used. This method was applied for the first time to this kind of problems in [1] and analysed in detail in [2]. According to this method the eigensolutions are looked for in the form $\exp(\lambda_i x)$ where the eigenvalues $\lambda_i \in \mathbb{C}(i = 1, \ldots, n)$ satisfy the characteristic equation $f(\lambda, a, Q, t, \ldots) = 0$ such that the general solution of the given equation is of the form $P_i(x) \exp \lambda_i x$. $P_i(x)$ are polynomials of degree $k_i < m_i - 1$ where m_i is the multiplicity of λ_i. Imposing the boundary conditions the neutral hypersurface, referred to as neutral (or secular) equation, is obtained in the form of a n-th order determinant $F(\lambda_i, \cos h\, \lambda_i l, \sin h\, \lambda_i l) = 0$, where l is a characteristic length.

Various sections of the neutral hypersurface corresponding to fixed values of all parameters but one may be determined by simultaneously solving the characteristic equation $f(\lambda, a, Q, T) = 0$ and the neutral equation written as $g(\lambda_i(a, Q, T)) = 0$.

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Although, generally, explicit forms for λ_i as functions of a, Q and T are not known, simplification may be deduced by examining the characteristic equation. The characteristic equation may be thought of as the implicit form of the function $\lambda \to \lambda(a, Q, T, \ldots)$ whose graph in the space (Re λ, Im λ, a, Q, $T, \ldots$) is a hypersurface with n sheets. For simple λ_i solutions of the characteristic equation (as in the case treated in [3]) this hypersurface has n distinct sheets. However, the presence of many parameters favour the existence of some hypersurface $T = T^*$ $(a, Q, \ldots)$ in the space of parameters such that at least one λ_i is multiple. In this case some sheets of the characteristic hypersurface $\lambda = \lambda(a, Q, T, \ldots)$ coalesce and, correspondingly, g vanishes identically. The points of the hypersurface $T = T^*$ are bifurcation points for the characteristic and neutral hypersurfaces: this is why the hypersurface $T = T^*$ is referred to as the bifurcation hypersurface.

The presented method assumes two advantages: (1) it involves the most reduced numerical calculations if referred to all other method. Indeed, the neutral equation is exact, it has a finite number of terms, this number is small and the terms are expressed as products of elementary (hyperbolic sine and cosine) functions; (2) it puts into evidence some bounds of the neutral hypersurfaces, shown for the first time in hydromagnetic stability theory context in [4]. These bounds are just the bifurcation hypersurfaces of the characteristic and neutral equations. As these bifurcation hypersurfaces are crossed, the mathematical properties of the solutions of the characteristic equation and the physical properties related to the neutral hypersurface qualitatively change. These phenomena are due to the occurrence of the set of parameters. Analogous bifurcation surfaces can be obtained in other stability problems too [5].

In this paper the analysis in [3] is pushed further, allowing for multiple λ_i for a problem in hydromagnetic stability. The geometrical interpretation of the characteristic equation and of the bifurcation surface is given in section 2. In section 3 various forms of the neutral equation are deduced. A bifurcation curve, representing the intersection of the neutral and bifurcation surfaces, is put into evidence in section 4. The last section summarizes the results concerning the neutral surface.

2. **Characteristic hypersurfaces and their associated bifurcation manifolds.** The linear neutral stability of the Couette flow of an electrically conducting fluid subjected to an axial magnetic field is governed by the following eigenvalue problem [2].

(2.1) $$\{(D^2 - a^2)^2 + Qa^2\}^2 v = - Ta^2(D^2 - a^2)\, v, \quad x \in (-0.5,\ 0.5),$$

(2.2) $$Dv = (D^2 - a^2)\, v = \{(D^2 - a^2)^2 + Qa^2\}\, v = D\{(D^2 - a^2)^2 + Qa^2\}\, v = 0 \ \text{ at } x = \pm 0{,}5.$$

where $T > 0$ is the Taylor number, $Q > 0$ is a measure of the magnetic field strength, $a > 0$ stands for the wavenumber and $v \in C^\infty[-0.5, 0.5]$. $v : [-0.5, 0.5] \to \mathbb{R}$. The smallest eigenvalue $T = T(a, Q)$ separates the domain of stability from that of instability. This eigenvalue is the solution of the neutral equation obtained by imposing to the general solution of (2.1) to satisfy (2.2). The general solution of (2.1) is expressed by using the solutions $\lambda_i(a, Q, T)$ of the corresponding characteristic equation

(2.3) $$(\lambda^2 - a^2)^4 + 2Qa^2(\lambda^2 - a^2)^2 + Ta^2(\lambda^2 - a^2) + Q^2a^4 = 0,$$

which, letting $\mu = \lambda^2 - a^2$, can be written as

(2.3)′ $$\mu^4 + 2\,Qa^2\mu^2 + Ta^2\mu + Q^2a^4 = 0.$$

The equation (2.3)′ represents a hypersurface in the 5-th dimensional space (Re μ, Imμ, a, Q, T) which for $T \neq T^* \equiv 16\ aQ\sqrt{Q}\,(3\sqrt{3})^{-1}$ has four sheets while for $T = T^*$, $a \neq 0$, $Q \neq 0$ it has three sheets. In the three-dimensional space (a, Q, T) of the parameters, the surface $T = T^*$, at whose points two sheets of the characteristic hypersurface are linked, represents the bifurcation (catastrophe) surface C for the set of the solution of the equation $F(\mu, a, Q, T) = \mu^4 + 2Qa^2\mu^2 + Ta^2\mu + Q^2a^4 = 0$. A qualitative illustration of C may be found in Fig. 1. The sections

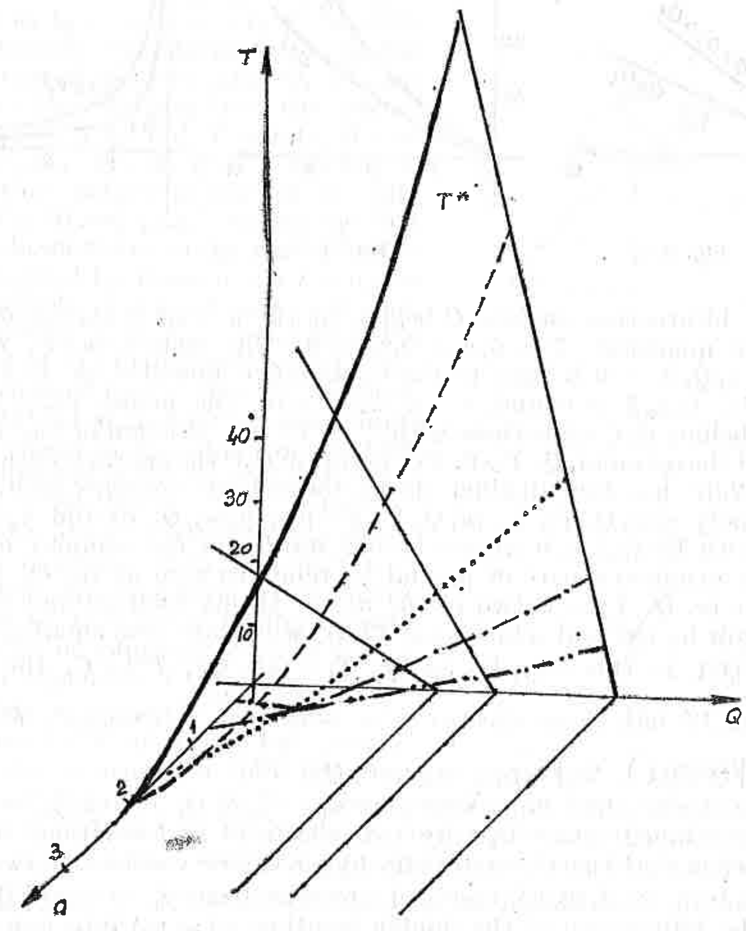

Fig. 1

through C by planes $Q = \text{const}$, are straightlines passing through $(0,0)$; they tend towards the a-axis as $Q \to 0$ and towards T-axis as $Q \to \infty$ (Fig. 2). The intersection $C \cap \{a = \text{const.}\}$ is a cusp which, for $a = 0$, degenerates into the axis $T = 0$ and goes further and further away from the Q-axis as a increases (Fig. 3). Finally, the sections of C by planes $T = \text{const}$, are hyperbole-like curves degenerating into the axis $a = 0$ and $Q = 0$ as $T \to 0$ and are further and further away from these axis as $T \to \infty$ (Fig. 4). The surface C is defined for $a, Q, T \geqslant 0$, hence it includes the a and Q axes.

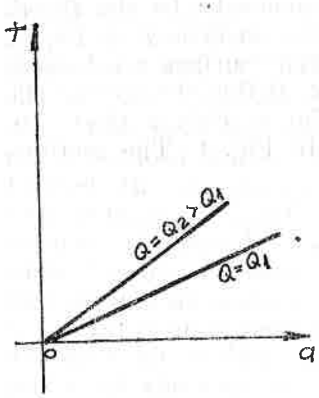

Fig. 2

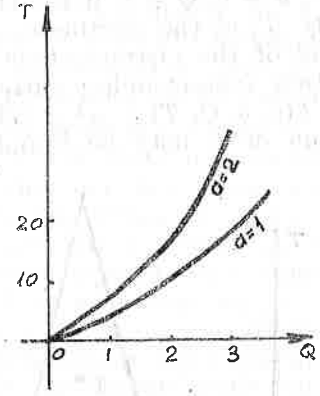

Fig. 3

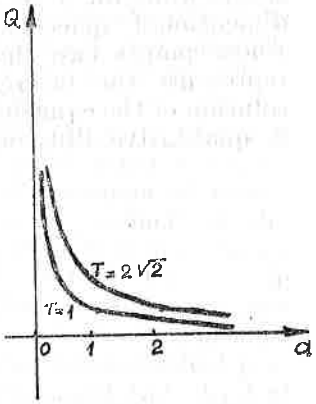

Fig. 4

The bifurcation surface C begins at the a- and Q-axes and covers the entire quadrant $T = 0, a > 0, Q > 0$. The points (a, Q, T) with $T > T^*, a, Q, T > 0$ belong to the domain O_1C bounded by C and the planes $Q = 0, a, T > 0$ and $a = 0, T, Q > 0$. The points (a, Q, T) with $T = T^*$ belong to C while those with $T < T^*$ are situated in the remainder O_2C of the octant $a, Q, T \geqslant 0$. For points of O_2C the characteristic hypersurface (2.3)$'$ has four distinct sheets formed by complex solutions of (2.3)$'$ namely $\mu_1(a, Q, T)$, $\mu_2(a, Q, T)$ $(= \bar{\mu}_1)$, $\mu_3(a, Q, T)$ and $\mu_4(a, Q, T)$ $(= \bar{\mu}_3)$, with $\mathrm{Re}\,(\mu_1) < 0$ where the bar stands for the complex conjugation. The imaginary part of μ_1 and μ_2 tends to zero as $(a, Q, T) \to C$. Hence, as $(a, Q, T) \to C$ two of the above sheets coalesce and they are formed now by the real solutions of (2.3)$'$ which are also equal $\mu_1 = \mu_2 = -a\sqrt{Q/3}$. In this way, for $(a, Q, T) = (a^*, Q^*, T^*) \in C$, the hypersurface (2.3)$'$ has three sheets: $\mu_1 = -a^*\sqrt{\dfrac{Q^*}{3}}$, $\mu_3 = \mu_3(a^*, T^*, Q^*) = -a^*\sqrt{Q^*/3}(1 - 2i\sqrt{2})$, $\mu_4 = \bar{\mu}_3(a^*, Q^*, T^*)$.

The sheets whose equations were $\mu_1 = \mu_1(a, Q, T)$ and $\mu_2 = \bar{\mu}_1(a, Q, T)$ are continued inside O_1C by two sheets of real solutions of (2.3)$'$. Hence inside O_1C the characteristic hypersurface consists of two sheets of real solutions of (2.3)$'$, emerged upwards from $\mu_1 = -a^*\sqrt{Q^*/3}$ (implying the bifurcation of the double solution $\mu_1 (= \mu_2)$ into two distinct real solutions μ_1 and μ_2), and two sheets of complex conjugate solutions

$\mu_3 = \mu_3(a, Q, T)$ and $\mu_4 = \bar{\mu}_3(a, Q, T)$. The following limiting situations special analysis deserve : $a = 0$, $Q = 0$, $T = 0$, corresponding to points (a, Q, T) situated on planes QOT, aOT and aOQ respectively and describing the meaningless physical situation of vanishing wavenumber, the degenerated case of the absence of the electrical effects and the degenerated case of the absence of thermal effect respectively. If $a = 0$ and $Q, T \geqslant 0$ then $\lambda = 0$ is the unique solution of equation (2.3) and it has the multiplicity 8. Hence all sheets $\lambda_i(a, Q, T)$ and $\mu_i(a, Q, T)$ coincide with the hyperplane $\mathrm{Re}\,\lambda = 0$, $\mathrm{Im}\,\lambda = 0$, and, consequently, the points (O, Q, T) of the plane $a = 0$ where $Q, T \geqslant 0$ (including the origin) are bifurcation points for the characteristic hypersurface (2.3), (2.3)$'$. If $T = 0$, $a, Q > 0$ then $\mu_1 = \mu_2 = ia\sqrt{Q}$, $\mu_3 = \mu_4 = -ia\sqrt{Q}$, $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = \sqrt{a\sqrt{Q}\,\mathrm{i}}$, $\lambda_3 = \lambda_4 = \lambda_7 = \lambda_8 = -\sqrt{a\sqrt{Q}\,\mathrm{i}}$. For $Q = 0$, $a, T > 0$ we have $\mu_1 = 0$, $\mu_{2,3,4} = -Ta^2\varepsilon_{2,3,4}$ where $\varepsilon^3_{2,3,4} = 1$. Finally for $Q = 0$, $T = 0$, $a > 0$ it follows $\mu_i = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = a$, $\lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = -a$. Analogous geometrical interpretation for the sheets $\lambda_i(a, Q, T)$ and $\mu_i(a, Q, T)$ may be done. Remark also that the planes $a = 0$ $(T, Q \geqslant 0)$ and $T = 0$ $(a, Q > 0)$ and the axis $T = 0$, $Q = 0$ consists of bifurcation points for the set of solutions of (2.3) and (2.3)$'$. Inside the first octant, in addition to the multiplicities of λ_i (implied by those of μ_i) λ_i may be also multiple if one of μ_i is $-a^2$ and, consequently $\lambda_i = \lambda_{i+4} = 0$; this takes place for (a, Q, T) belonging to the surface C_1 defined by $T = (Q + a^2)^2$, $a, Q > 0$. The points of C_1 are bifurcation points for the sheets of (2.3) but not of (2.3)$'$. In this discussion the form $\lambda^8 - 4a^2\lambda^6 + (6a^4 + 2Qa^2)\lambda^4 + (Ta^2 - 4Qa^4 - 4a^6)\lambda^2 - a^4[T - (Q^2 + 2a^2Q + a^4)] = 0$ of equation (2.3) was useful. A qualitative graph of bifurcation corresponding to C is given in Fig. 5 while the nature of solutions of (2.3)$'$ is analysed in Appendix 1.

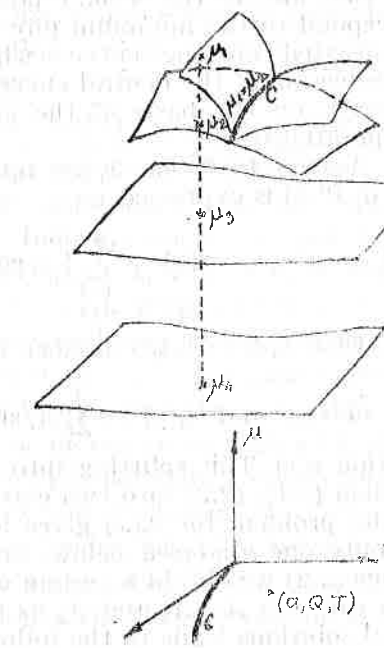

Fig. 5

This comment emphasizes also the change of mathematical properties of the solutions of the characteristic equation (2.3)$'$ as a bifurcation manifold in the (a, Q, T) plane is crossed. In fact, the characteristic equation is (2.3) but the above discussion may be analogously carried out for this equation to which a 8-sheeted characteristic hypersurface in the $(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda, a, Q, T)$-space corresponds; for points of $C \setminus C_1$, (with $a, Q, T > 0$) four or two of these sheets coalesce according to the case $a^* = \sqrt{Q^*/3}$ or $a^* \neq \sqrt{Q^*/3}$. The characteristic hypersurface for (2.3) has also two coincident sheets for $T = (Q + a^2)^2$, $Q \neq 3a^2$ and 8 coincident sheets for $a = 0$.

3. The neutral equations. The form of the general solution of equation (2.1) and, consequently, of the associated secular equation, is different in different regions of the first octant $a, Q, T \geqslant 0$, according to the multiplicity m_i of the roots λ_i of the characteristic equation (2.3) and, therefore, of the roots μ_i of (2.3)'. Since these multiplicities occur for points (a, Q, T) situated on the bifurcation manifold B consisting of the points of C, C_1, and the coordinate planes, it follows that the boundaries of these regions belong to B. In the sequel the secular equation will be derived in each such region or portion of B (consisting of points enjoying the property that the multiplicities m_i remains the same for all these points). Since the points of B are solutions of the secular equations extended to the clo ure of their domain of definition, this procedure enables the selection of the secular points which belong to B. Remark also that among the secular points there are the neutral points which correspond to the minimum physical parameters Q and T and therefore are situated on the curves which are close to the Q-and T-axes. The selection of the neutral curves, and, generally, the neutral manifolds proceeds, on the basis of the graphical representation (subsequent to computations).

Assume first that λ_i are mutually distinct. Then the general solution of (2.1) is expressed as

$$v(x) = \sum_{i=1}^{4} (A_i \cosh \lambda_i x + B_i \sinh \lambda_i x) \tag{3.1}$$

such that $v(x) = v_e(x) + v_0(x)$ where $v_e(x) = \sum_{i=1}^{4} A_i \cosh \lambda_i x$ is the even part of $v(x)$ and $v_0(x) = \sum_{i=1}^{4} A_i \sinh \lambda_i x$ represents the odd part of the solution $v(x)$. This splitting into even and odd part allows a splitting of problem (2.1), (2.2) into two correspondig problems for $v_e(x)$ and $v_0(x)$. As the problem for $v_e(x)$ gives lower stability bounds, this problem is the only one analysed below. So, imposing to $v_e(x)$ the boundary conditions (2.2) we obtain a system of 4 linear homogenous algebraic equations with A_1, A_2, A_3 and A_4 as unknowns. The condition to have nontrivial solutions leads to the following secular equation [2].

$$\begin{vmatrix} \lambda_1 \sinh(\lambda_1/2) \dots \lambda_4 \sinh(\lambda_4/2) \\ \mu_1 \cosh(\lambda_1/2) \dots \mu_4 \cosh(\lambda_4/2) \\ (\mu_1^2 + Qa^2) \cosh(\lambda_1/2) \dots (\mu_4^2 + Qa^2) \cosh(\lambda_4/2) \\ \lambda_1(\mu_1^2 + Qa^2) \sinh(\lambda_1/2) \dots \lambda_4(\mu^2 + Qa^2) \sinh(\lambda_4/2) \end{vmatrix} = 0, \ a, Q, T \geqslant 0 \quad (a, Q, T) \notin C \cup C_1. \tag{3.2}$$

This equation is of the form $g(\lambda_1(a, Q, T), \dots, \lambda_4(a, Q, T))$ and describe, in the region $\{(a, Q, T) \in C \cup C_1 \mid a, Q, T \geqslant 0\}$ of the (a, Q, T) parameter space, a part $\mathcal{S}_1$ of the complete secular surface $\mathcal{S}$. Thus, the

determination of $\mathcal{S}_1$ (which is to be expected to consist of many curves) follows by simultaneously solving (2.3) and (3.2) or, equivalently, (2.3)' and (3.2)' as was shown in [3]. Generally, it is not necessary to know explicit expressions $\lambda_i = \lambda_i(a, Q, T)$. (In the case of problem (2.1), (2.2) these expressions are given in Appendix 2).

Let us remark that this procedure to determine the secular surface $\mathcal{S}_1$ must be applied only to these points (a, Q, T) for which λ_i are distinct i.e. for $(a, Q, T) \notin C \cup C_1$. Equation (3.2) is fundamental. All the other secular equations in other regions of the first octant will be related to (3.2): they will be appropriate limits of (3.2). Although the points of physical interest are not situated in the coordinate planes, their analysis is imposed by continuation reasons, implied by these limites. Recall that bifurcation points of a set do not necessary belong to that set but generally are limit points of that set.

Assume now that some of μ_i are not simple; this is possible [3] only if $(a, Q, T) \in C$ where C is the bifurcation surface consisting of points (a^*, Q^*, T^*) with $T^* = 16\, a^* Q^* \sqrt{Q^*}(3\sqrt{3})^{-1}$.

In this case, for $a \neq 0$, $Q \neq 0$, $\mu_1 = \mu_2 = -a^*\sqrt{Q^*/3}$, $\mu_{3,4} = a^*\sqrt{Q^*/3}(1 \pm 2i\sqrt{2})$, $\lambda_1 = \sqrt{a^{*2} - a^*\sqrt{Q^*/3}}$, $\lambda_2 = \bar{\lambda}_1$, $\lambda_{3,4} = \sqrt{a^{*2} + a^*\sqrt{Q^*/3}(1 \pm 2i\sqrt{2})}$, $\lambda_{4+i} = -\lambda_i$, $i = 1, 2, 3, 4$. These expressions show that the equality $\mu_1 = \mu_2$ does not always imply $\lambda_1 = \lambda_2$. Indeed, even if $\mu_1 (= \mu_2)$ is real, λ_1 may be real and positive (and equal to λ_2), or $\lambda_1 = \lambda_2 = 0$, or, finally, λ_1 is imaginary (and distinct from $\lambda_2 = -\lambda_1$). However, if $\mu_1 = \mu_2$ it follows that the appropriately extended secular equation (3.2) vanishes because either $\lambda_1 = \lambda_2 \neq 0$, or $\lambda_1 = \lambda_2 = 0$, or $\lambda_1 = -\lambda_2$ and hence the first two columns in (3.2) are identical. Assume first that equation (3.2) was extended by supplying the domain of definition of (3.2) with points of $C \setminus C_1$ (and for $a > 0$) which eliminates the possibility of vanishing λ_i. So, for every point (a^*, T^*, Q^*) with $a^* > 0$ of the surface $C \setminus C_1$ (3.2) vanishes; but this does not means that all points of $C \setminus C_1$ belong to the neutral surface. The reason is that for those points of $C \setminus C_1$ for which $\lambda_1 = \lambda_2$ (and hence $\lambda_5 = \lambda_6$) the expression of the general solution of (2.1) (which is no longer (3.1)) reads as

$$v(x) = (A_1 + A_2 x) \cosh \lambda_1 x + (B_1 + B_2 x) \sinh \lambda_1 x + \sum_{i=3,4} (A_i \cosh \lambda_i x + B_i \sinh \lambda_i x) \tag{3.3}$$

and, correspondingly, its even part expresses as $v_e(x) = A_1 \cosh \lambda_1 x + B_2 x \sinh \lambda_1 x + A_3 \cosh \lambda_3 x + A_4 \cosh \lambda_4 x$. Then a reasoning, similar to that which led to (3.2), enables us to express the secular equation $\mathcal{S}_2$ in the region $\{(a, Q, T) \in C \setminus C_1 \mid a > 0, Q > 0\}$ of the parameter space, in the form

$$D^*=\begin{vmatrix} \lambda_1 \sinh(\lambda_1/2) & \sinh(\lambda_1/2)+(\lambda_1/2)\cosh(\lambda_1/2) \\ \mu_1 \cosh(\lambda_1/2) & 2\lambda_1\cosh(\lambda_1/2)+(\mu_1/2)\sinh(\lambda_1/2) \\ (\mu_1^2+Qa^2)\cosh(\lambda_1/2) & 4\lambda_1\mu_1\cosh(\lambda_1/2)+[(\mu_1^2+Qa^2)/2]\sinh(\lambda_1/2) \\ \lambda_1(\mu_1^2+Qa^2)\sinh(\lambda_1/2) & (4\lambda_1^2\mu_1+\mu_1^2+Qa^2]\sinh(\lambda_1/2)+(\lambda_1(\mu_1^2+Qa^2)/2)\cosh(\lambda_1/2) \end{vmatrix}$$

(3.4)

$$\begin{matrix} \lambda_3\sinh(\lambda_3/2) & \lambda_4\sinh(\lambda_4/2) \\ \mu_3\cosh(\lambda_3/2) & \mu_4\cosh(\lambda_4/2) \\ (\mu_3^2+Qa^2)\cosh(\lambda_3/2) & (\mu_4+Qa^2)\cosh(\lambda_4/2) \\ \lambda_3(\mu_3^2+Qa^2)\sinh(\lambda_3/2) & \lambda_4(\mu_4^2+Qa^2)\sinh(\lambda_4/2) \end{matrix}\Bigg| = 0,\ a>0;\ Q>0,\ (a,Q,T)\in C\setminus C_1.$$

Formally, this equation may be obtained from (3.2) by simple differentiation of the second column and subsequently replacing λ_2 by λ_1; this is natural since (3.3) is obtained from (3.1) in the same way. Expression (3.3) may be deduced also by writing (3.1) as (3.1)′

$$\begin{aligned} v(x) &= A_1(\cosh\lambda_1 x-\cosh\lambda_2 x)+(A_2+A_1)\cosh\lambda_2 x+ \\ &+B_1(\sinh\lambda_1 x-\sinh\lambda_2 x)+(B_2+B_1)\sinh\lambda_2 x+ \\ &+\sum_{i=3,4}(A_i\cosh\lambda_i x+B_i\sinh\lambda_i x) \end{aligned} \qquad (3.1)'$$

or, equivalently, as

$$\begin{aligned} v(x) &= B_2' x\frac{\cos\lambda_1 x-\cos\lambda_2 x}{(\lambda_1-\lambda_2)\,x}+A_1'\cosh\lambda_2 x+ \\ &+A_2' x\frac{\sinh\lambda_1 x-\sinh\lambda_2 x}{(\lambda_1-\lambda_2)\,x}+B_1'\sinh\lambda_2 x+ \\ &+\sum_{i=3,4}(A_i\cosh\lambda_i x+B_i\sinh\lambda_i x) \end{aligned} \qquad (3.1)''$$

where $A_1'=A_1(\lambda_1-\lambda_2)$, $B_2'=A_2+A_1$, $B_1'=B_1(\lambda_1-\lambda_2)$ and $A_2'=B_2+B_1$ and then letting λ_2 to tend to λ_1. Assuming that A_1', A_2', B_2' remain constant, in the limit (3.1)″ turns to (3.3) (of course, dropping the primes). Similarly if in (3.2) the first column is substracted from the second, the resulted column is divided by $(\lambda_1-\lambda_2)/2$ and then λ_2 is let to tend to λ_1, one obtain (3.4). The two ways of obtaining (3.3) and (3.4) are equivalent because

$$\lim_{\lambda_2\to\lambda_1}\frac{\cos\lambda_1 x-\cos\lambda_2 x}{(\lambda_1-\lambda_2)\,x}=\frac{\partial\cos\lambda_2 x}{\partial\lambda_2}\Bigg|_{\lambda_2=\lambda_1}$$

and show that for $T<T^*$ (i.e. when all λ_i are complex) equation (3.2) is defined by a real function,

for $T>T^*$ this is an imaginary function while for $T=T^*$ the mentioned function is real or imaginary according to whether λ_1 is a purely imaginary or real number.

A third method to derive (3.4) from (3.2) is based on the asymptotic expansions of $\mu_{1,2}$ and $\lambda_{1,2}$ with respect to the asymptotic sequence $\{1, \varepsilon, \varepsilon^2, \dots\}$ for $\varepsilon\to 0$. Here ε is a small (i.e. $|\varepsilon|$ is small) complex number of the order of magnitude of the $\lambda_1-\lambda_1^*$, where * indicates the values on the surface $T=T^*$. Thus for $\varepsilon\to 0$, $\mu_1=\mu_1^*+2\varepsilon\lambda_1^*$,

$$\lambda_1=\sqrt{\mu_1^*+2\varepsilon\lambda_1^*+a^2}=\sqrt{\mu_1^2+a^2}\left(1+\frac{\varepsilon\lambda_1^*}{\mu_1^*+q^2}+\dots\right)=\lambda_1^*+\varepsilon+\dots,$$

$\mu_2=\mu_2^*+2\bar\varepsilon\lambda_1^*$, $\lambda_2=\lambda_1^*+\bar\varepsilon+\dots$ if λ_1^* is real and $\lambda_2=-\lambda_1^*+\bar\varepsilon$ if λ_1^* is imaginary. Then, for $\lambda_1^*\in\mathbb{R}$ (i.e. for points (a, Q) with $a>\sqrt{Q/3}$) taking into account that $\sinh(x+\varepsilon/2)\sim\sinh x+\varepsilon\cosh x+(\varepsilon^2/2!)\sinh x+\dots$, the first column in (3.2) is expressed as

$$\lambda_1^*+\sinh\frac{\lambda_1^*}{2}+\varepsilon\left(\sinh\frac{\lambda_1^*}{2}+\frac{\lambda_1^*}{2}\cosh\frac{\lambda_1^*}{2}\right)+\dots$$

$$\mu_1^*\cosh\frac{\lambda_1^*}{2}+\varepsilon\left(2\lambda_1^*\cosh\frac{\lambda_1^*}{2}+\frac{\mu_1^*}{2}\sinh\frac{\lambda_1^*}{2}\right)+\dots$$

$$(\mu_1^{*2}+Qa^4)\cosh\frac{\lambda_1^*}{2}+\varepsilon\left(4\lambda_1^*\mu_1^*\cosh\frac{\lambda_1^*}{2}+\frac{\mu_1^{*2}+Qa^2}{2}\sinh\frac{\lambda_1^*}{2}\right)+\dots$$

$$\lambda_1^*(\mu_1^{*2}+Qa^2)\sinh\frac{\lambda_1^*}{2}+\varepsilon\left[(\mu_1^{*2}+Qa^2+4\lambda_1^{*2}\mu_1^*)\sinh\frac{\lambda_1^*}{2}+\frac{\lambda_1^*}{2}\left(\mu_1^{*2}+Qa^2\right)\cosh\frac{\lambda_1^*}{2}\right)+\dots$$

while the second column is obtained from the first one replacing ε by $\bar\varepsilon$. These expressions show that the leading term of the asymptotic expansion of (3.2) with respect to the sequence $\{1, \varepsilon, \varepsilon^2, \dots\}$ for $\varepsilon\to 0$ is just (3.4) (obtained, equivalently, substracting the first column from the second, dividing the result by $\bar\varepsilon-\varepsilon$ and letting $\varepsilon\to 0$), i.e. $D\sim(\bar\varepsilon-\varepsilon)D^*+\dots$ where $D=0$ and $D^*=0$ represent equations (3.2) and (3.4) respectively. Similarly, if λ_1^* is imaginary, it follows that $D\sim-(\varepsilon+\bar\varepsilon)D^*+\dots$ such that the first approximation leads also to (3.4).

The second approach points out that (3.4) was obtained from (3.2) by operations invariating the solutions of (3.2) followed by a passage to the limit. Hence (3.4) is a limit of (3.2) as $\lambda_2\to\lambda_1$. In this case $T\to T^*$ such that, replacing T^* by its expression, equation (3.4) contains only two unknowns: a and Q. Denote by S_{21} the curve (or curves) consisting of points (a, Q), solutions of (3.4), and let $\mathscr{S}_2$ be the curve (or curves) consisting of points (a, Q, T^*) where $(a, Q)\in\mathscr{S}_{21}$. Hence $\mathscr{S}_{21}$ is the orthogonal projection of $\mathscr{S}_2$ on the (a, Q) plane. The second approach shows that $\mathscr{S}_2$ is a bound of the secular surface $\mathscr{S}_1$ consisting of limit points of some or all curves of $\mathscr{S}_1$ and realizes extrema of these curves. (This extremum property is realised only for curves of $\mathfrak{S}_1$ and not of the whole secular surface $\mathfrak{S}$). $\mathfrak{S}_2$ is itself neutral. As $\mathfrak{S}_2$ is a neutral curve belonging to $C\setminus C_1$ it means that it is a bifurcation curve of $C\setminus C_1$ and $\mathfrak{S}$ and, hence, of $C\setminus C_1$ and E, where E is defined as the surface consisting of points (a, Q, T) of the first octant satisfying equation (3.2).

If the domain of definition of (3.4) is extended to the projection of C on the (a, Q) plane for $a, Q > 0$, then the secular curve $\mathfrak{S}_2$ will correspond only to a part of the solutions of this extended equation. Indeed as the point (a, Q) crosses the curve $a = \sqrt{Q/3}$, from imaginary λ_1 becomes real such that D^* changes from real to imaginary and (3.4) holds and consequently the curve $a = \sqrt{Q/3}$ is also a solution curve of (3.4). However, not all points of the curve $a = \sqrt{Q/3}$ (i.e. $C \cap C_1$) with a, $Q > 0$ are secular. To separate them we take into account that, in this case, $\mu_1 = \mu_2 = -a^2$, $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$, $Q = 3a^2$, $T = 16a^4$, $\lambda_{3,7} = \pm a\sqrt{2 + 2i\sqrt{2}}$, $\lambda_{4,8} = \pm a\sqrt{2 - 2i\sqrt{2}}$ such that the corresponding general solution of (2.1), (2.2) is

$$v = A_1 \cdot 1 + B_1 \cdot x + A_2 \cdot x^2 + B_2 \cdot x^3 + \sum_{i=3}^{4} (A_i \cosh \lambda_i x + B_i \sinh \lambda_i x) \tag{3.5}$$

and has even part

$$v_e = A_1 + A_2 x^2 + A_3 \cosh \lambda_3 x + A_4 \cosh \lambda_4 x \tag{3.6}$$

such that, denoting by $P(C \cap C_1)$ the projection of $(C \cap C_1)$ on the a, Q plane, the corresponding secular equation is expressed as

$$\begin{vmatrix} 0 & 1 & \lambda_3 \sinh(\lambda_3/2) & \lambda_4 \sinh(\lambda_4/2) \\ -a^2 & 2 - a^2/4 & \mu_3 \cosh(\lambda_3/2) & \mu_4 \cosh(\lambda_4/2) \\ 4a^4 & a^4 - 4a^2 & (\mu_3^2 + Qa^2)\cosh(\lambda_3/2) & (\mu_4^2 + Qa^2 \cosh(\lambda_4/2) \\ 0 & 4a^4 & \lambda_3(\mu_3^2 + Qa^2)\sinh(\lambda_3/2) & \lambda_4(\mu_4^2 + Qa^2)\sinh(\lambda_4/2) \end{vmatrix} = 0, \tag{3.7}$$

for $(a, Q) \in P(C \cap C_1)$, a, $Q > 0$.

This equation may be also deduced from (3.4) by the above mentioned approaches. Hence (3.7) may be derived from (3.2) by differentiating the second column two times and then letting $T \to T^*$ and $a \to \sqrt{Q/3}$. In Appendix 3 it is proved that (in the domain $C \cap C_1$, $a, Q > 0$) it has no solution and, consequently, there is no secular point in that domain. Hence none of the points of the curve $a = \sqrt{Q/3}$ is secular. In exchange, if a is allowed to take the value $a = 0$ (which implies also $Q = 0$) then $a = 0$ is the solution of the corresponding extended equation. Since $(a, Q) = (0,0)$ is the common solution of (3.4) extended to $P(C\setminus C_1) \cup \{0,0\}$ and of (3.7) extended to $P(C \cap C_1)$ it follows that this point is a bifurcation point for the set of solutions of (3.4) extended to

$C\setminus C_1 \cup \{0, 0\}$. Indeed, as is it shown by computations, it is a limit point for *all* the solution curves of (3.4) and, at the same time, it belongs to the curve $a = \sqrt{Q/3}$ (defined for $a, Q \geqslant 0$) (fig. 6).

Let us see now which among points $(0, Q, T)$, Q, $T \geqslant 0$ are secular. In this case $\lambda_i = 0$, $i = \overline{1, 8}$, $\mu_i = 0$, $i = \overline{1,4}$ such that the general solution of (2.1) may be written as

$$v = \sum_{i=1}^{8} A_i x^{i-1} \tag{3.8}$$

whose even part is

$$v_e = A_1 + A_2 x^2 + A_4 x^4 + A_6 x^6, \tag{3.9}$$

which leads to the secular equation whose first column has only vanishing entries and consequently all points $(0, Q, T)$ with $Q, T \geqslant 0$ are secular.

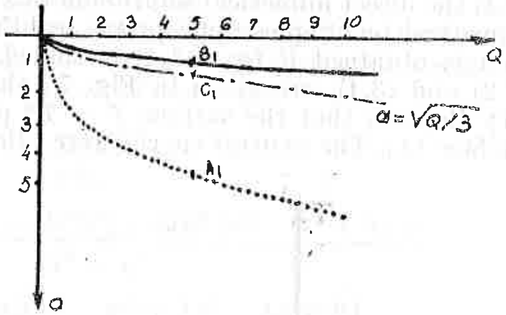

Fig. 6

These points satisfy the appropriately extended equation (3.2) and, consequently, are bifurcation points for E.

For points $(a, 0, T)$, a, $T > 0$ $T \neq a^4$ in the plane $Q = 0$ we have $\lambda_{1,2} = \pm a$, $\lambda_3, \ldots, {}_8 = \pm \sqrt{a^2 - \sqrt[3]{Ta^2}\,\varepsilon}$ where $\varepsilon^3 = 1$ and the neutral equation (3.2) must be considered.

Consider now the points $(a, Q, T) \in C_1$, $a > 0$ $a \neq \sqrt{Q/3}$; in this case the general solution of (2.1), (2.2) is $v = A_1 + B_1 x + \sum_{i=2}^{4} A_i \cosh \frac{\lambda_i}{2} + B_i \sinh \frac{\lambda_i}{2}$ and has the even part $v_e = A_1 + \sum_{i=2}^{4} A_i \cosh \frac{\lambda_i}{2}$. This v_e may be obtained from the even part of (3.1) letting $\lambda_1 = 0$ and consequently the corresponding neutral equation is obtained from (3.2) for $\lambda_1 = 0$. On the basis of computations we conjecture that no point $(a, Q, T) \in C_1$, $a > 0$, $a \neq \sqrt{Q/3}$ is secular.

Last, let us take $(a, Q\ T) \in C_1$, $a = \sqrt{Q/3} > 0$. In this case (2.3) becomes equation

$$\lambda^8 - 4a^2 \lambda^6 + 12a^4 \lambda^4 = 0, \tag{2.3)''$$

whose solutions are $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$, $\lambda_{3,7} = \pm a\sqrt{2}\sqrt{1 + i\sqrt{2}}$, $\lambda_{4,8} = \pm a\sqrt{2}\sqrt{1 - i\sqrt{2}}$. This case, leading to (3.7), was examined above and led to the conclusion that no point $(a, Q, T) \in C_1$ with $a = \sqrt{Q/3} > 0$ is neutral.

It follows that the secular manifold $\mathfrak{S}$ consists of: points $(a, Q, T) \notin C \cup C_1$, $a, Q, T > 0$ satisfying (3.2) and situated on $\mathfrak{S}_1$; points $(a, Q, T) \in C\setminus C_1$ a, Q, $T > 0$, $T = T^*$ $(\neq 16a^4)$ where (a, Q) satisfy (3.4) (and correspondingly (a, Q, T^*) are limit points of (3.2) for $T \to T^*$) and are situated on $\mathcal{S}_2$; all the points of the plane $a = 0$ (including Q- and T-axes and the origin $(0, 0, 0)$); all the points of the plane $Q = 0$, but $T \neq a^4$ satisfying (3.2). Since some of these points are bifurcation points for E and some of the projection on the (a, Q) plane are bifurcation points for (3.4) the direct numerical solution of (3.2) and (3.4) may be performed by numerical techniques appropriate to bifurcation. A reduction of calculation is obtained if first (3.4) is solved. Some of our results, based on (3.2) and (3.4), are given in Fig. 7; they agree perfectly with those in [6]. Remark that the surface $T = T^*$ provides a bound for the neutral surface [4]. The neutral curves were plotted by continuous lines.

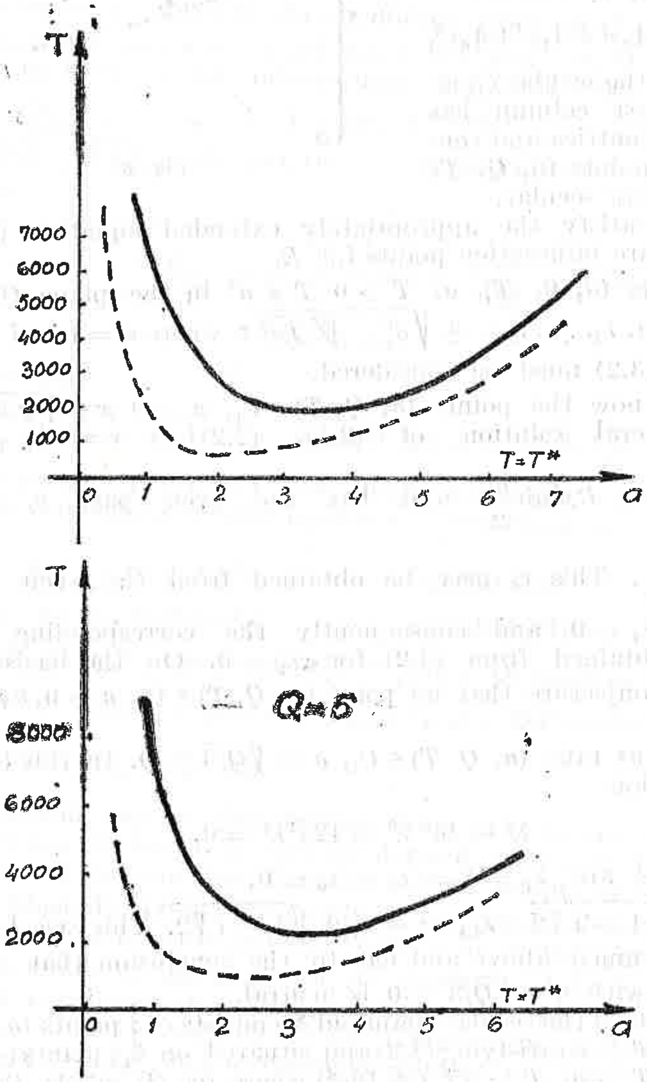

Fig. 7a and b

APPENDICES

1. Properties of the solutions of equation (2.3)′. All the real solutions μ of (2.3)′ are negative since from (2.3)′ it follows $\mu = -(\mu^2 + Qa^2)^2 T^{-1} a^{-2}. < 0$. It follows that (2.3)′ cannot have only real solutions because their sum (i.e. the coefficient of μ^3 in (2.3)′) is zero.

For $T > T^*$ equation (2.3)′ has no real solutions. Indeed, assume that the solution of (2.3)′ is $\mu = \sqrt{Qa^2}\,(x + iy)$, $x, y \in \mathbb{R}$ (hence $y = 0$ correspond to real solutions) such that from (2.3)′ we have

$$\text{(A.1)} \qquad T = -\frac{(\mu^2 + a^2Q)^2}{a^2\mu} =$$

$$= -aQ\sqrt{Q}\left\{\frac{x[(x^2 - y^2 + 1)^2 - 4x^2y^2] + 4xy^2(x^2 - y^2 + 1) +}{x^2 + y^2} + \frac{i\{4x^2y(x^2 - y^2 + 1) - y[(x^2 - y^2 + 1)^2 - 4x^2y^2]\}}{x^2 + y^2}\right\}.$$

Since T is a real number it follows that $4x^2y(x^2 - y^2 + 1) - y[(x^2 - y^2 + 1)^2 - 4x^2y^2] = 0$ which implies either $y = 0$ or

$$\text{(A.2)} \qquad 4x^2(x^2 - y^2 + 1) - [(x^2 - y^2 + 1)^2 - 4x^2y^2] = 0$$

where $y = 0$ or $y \neq 0$. Taking into acount (A.2) in (A.1) we obtain

$$\text{(A.1)}' \qquad T = -4aQ\sqrt{Q}\,x(x^2 - y^2 + 1).$$

As $T > 0$ it follows that $x(x^2 - y^2 + 1) < 0$ and, consequently, from (A.2) we have the equality

$$\text{(A.2)}' \qquad 2x\sqrt{x^2 + 1} = -(x^2 - y^2 + 1),$$

valid for $y = 0$ as well as for $y \neq 0$. If (A.2)′ is accounted for in (A.1) we get

$$\text{(A.1)}'' \qquad T = 8aQ\sqrt{Q}\,x^2\sqrt{x^2 + 1}.$$

If, now, we take $y = 0$ in (A.1) we obtain

$$\text{(A.1)}''' \qquad T = aQ\sqrt{Q}\,(x^2 + 1)^2/|x|.$$

This expression is valid for every real solution $\mu = x\sqrt{Qa^2}$ of (2.3)′ while the only real solution of (2.3)′ involved in (A.2) and, consequently, in (A.1)″ is $\mu = x\sqrt{Qa^2}$ with $x = -\dfrac{1}{\sqrt{3}}$.

The solutions of (2.3)′ fall into three cases: (a) two solutions are real and distinct $\mu_{1,2} = \sqrt{Qa^2}\,x_{1,2} < 0$ and two are complex conjugate $\mu_{3,4} = \sqrt{Qa^2}\,r(\cos\theta + i\sin\theta)$. Writing $S = x_1 + x_2$, $P = x_1x_2$ the Viète relations are expressed as: $S + 2r\cos\theta = 0$, $P + r^2 + S\,2r\cos\theta = 2$, $P \cdot 2r\cos\theta + Sr^2 = -T(aQ\sqrt{Q})^{-1}$, $Pr^2 = 1$.The first of these four equations implies $\cos\,\theta > 0$, the third shows that $P < r^2$; solving the equations it follows $S = (1-r^2)/r$, $P = r^{-2}$ and hence $x_{1,2} = \frac{1-r^2}{2r} \pm \pm \frac{1}{2r}\sqrt{r^4 - 2r^2 - 3}$. As $x_{1,2}$ are real it is necessary that $r^4 - 2r^2 - 3 > 0$ therefore $r^2 > 3$. But $r^2 = x_3^2 + y_3^2 = x_4^2 + y_4^2$ where (x_3, y_3) and (x_4, y_4) satisfy (A.2)′.

Then taking into account (A.2)′ the condition $r^2 > 3$ implies $x_3 > 1/\sqrt{3}$ such that (A.1)″ gives $T > T^*$. Similar reasonings show that: (b) for $T < T^*$ equation (2.3)′ has four complex conjugate while for (c) $T = T^*$ this equation has two real and equal solutions, the other two solutions being complex conjugate. In the case (b) x_1 and x_2 will stand for the complex conjugate numbers $\mu_{1,2}/\sqrt{Qa^2}$ while in the case (c) they will be equal real numbers $(\mu_1/\sqrt{Qa^2} = \mu_2/\sqrt{Qa^2})$.

The above analysis could be carried out directly on the explicit form of the solution of (2.3)′, obtained by reducing the solution μ of this fourth degree equation $w = a\mu\sqrt{Q}$ to the solution of two second degree equations

$$w^2 + 0{,}5\,Aw + z - 16\,(3\sqrt{3}A)^{-1}\alpha = 0, \tag{A.3}$$

where $\alpha = TT^{*-1}$, $A = \pm 2\sqrt{2}\sqrt{z-1}$ and y is any real solution of the third degree equation

$$z^3 - z^2 - z + 1 - 32\,(27)^{-1}\,\alpha^2 = 0. \tag{A.4}$$

For $T > T^*$ i.e. $\alpha < 1$ the discriminant of (A.4) is $\left(\frac{16}{27}\right)^2 \alpha^2(\alpha^2 - 1) < 0$ hence the equation has three real solutions $Z_1 = \frac{1}{3} + u + v$, $Z_2 = \frac{1}{3} + \varepsilon u + \varepsilon^2 v$, $Z_3 = \frac{1}{3} + \varepsilon^2 u + \varepsilon v$ where $\varepsilon^3 = 1$, $u = \sqrt[3]{\frac{16}{27}}\sqrt[3]{\alpha^2 - 0.5 + \alpha\sqrt{\alpha^2 - 1}}$ and $v = \sqrt[3]{\frac{16}{27}}\sqrt[3]{\alpha^2 - 0.5 - \alpha\sqrt{\alpha^2 - 1}}$.

2. **Properties of solutions of (2.3).** In the paper we used the relations $\lambda_{1,5} = \pm\sqrt{\mu_1 + a^2}$, $\lambda_{2,6} = \pm\sqrt{\mu_2 + a^2}$, $\lambda_{3,7} = \pm\sqrt{\mu_3 + a^2}$, $\lambda_{4,8} = \pm \pm\sqrt{\mu_4 + a^2}$ where λ_i are the solutions of the characteristic equation (2.3). All multiplicities of μ_i carry correspondingly over the multiplicity of λ_i. Additional multiplicity of λ_i arise at those points (a, Q, T) for which $\lambda_i = 0$. This takes place if one among μ_i, say μ_j, is real and namely $\mu_1 = -a^2$, which implies $T = (Q + a^2)^2$. The surface

$C_1 = \{(a, Q, T) \in \mathbb{R}^3 \mid T = (Q + a^2)^2,\ a,\ Q > 0\}$ intersects the surface $T = T^*$ along a curve C_1^* whose projection on the (a, Q) plane is the parabola $a = \sqrt{Q/3}$, $a, Q > 0$. The points of $C_1 \setminus C_1^*$ are of bifurcation for the solutions of (2.3), two such sheets coincide and are $\lambda_1 = \lambda_5 = 0$. In fig. A2.1 we represented the intersection of the plane Q = constant with the surface C_1 (i.e. the curve $T = (Q + a^2)^2$) and with the surface C (i.e. the curve $T = T^*$). The last two curves intersect at the point $\mathfrak{P} = (\sqrt{Q/3},\ 16Q^2/9)$. Indeed $(Q + a^2)^2 = \frac{16}{3\sqrt{3}}\,Q\sqrt{Q}\,a$ implies $b^4 + 6b^2 - 16b + 9 = 0$ where $b = a/\sqrt{Q/3}$, therefore $(b-1)^2\,(b^2 + 2b + 9) = 0$ i.e. $b_{1,2} = 1$, hence $a_{1,2} = \sqrt{Q/3}$. The point $\mathfrak{P}$ coincides with the intersection of the curve C_1^* with the planes Q = constant. Our numerical calculations were carried out for various fixed values of Q; that is why we took into account that in planes Q = constant the nature of λ_i at a point (a, Q, T) depends on the position of that point with respect to the curves $T = (Q + a^2)^2$ and $T = T^*$. For instance below the straightline $T = T^*$ all λ_i are complex while above $T = T^*$ four among λ_i are complex and four are either real or two of them real and two imaginary. As the curve $T = (Q + a^2)^2$ is crossed the two λ_i which on the different sides of that curve were real and respectively imaginary vanish.

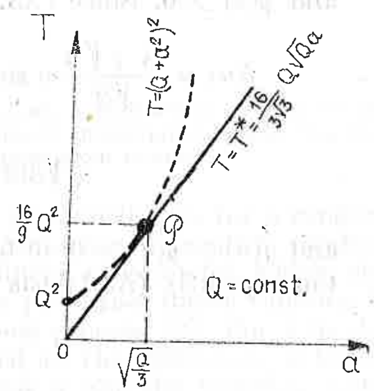

Fig. A2.1

3. **Equation (3.7.).** By simple algebra equation (3.7) becomes (A3.1) $t_{3r}^2 + t_{3i}^2 = 3\sqrt{2}\,a^2\,(\sqrt{2}\,t_{3r} + t_{3i})$, where $t_3 = \lambda_3 \tanh(\lambda_3/2) = t_{3r} + i\,t_{3i}$. Writing $\lambda_3 = u + iv$ where $u = a\sqrt{\sqrt{3} + 1}$, $v = a\sqrt{\sqrt{3} - 1}$, (A3.1) reads (A3.2) $(\sqrt{2}\cosh u - \cos v) - \sqrt{3}(\sqrt{2}u + v)\sinh u = \sqrt{3}\,(u - 2v)\sin v$. Since $v = (\sqrt{3} - 1)\,u/\sqrt{2}$, $h = \sqrt{3}\,(\sqrt{2}u + v)\sinh u - (\sqrt{2}\cosh u - \cos v)$ is a function of u alone; we have $h(0) = 0$, $\frac{dh}{du} = l(u) + \frac{3 + \sqrt{3}}{\sqrt{2}}\,u\cosh u$ where $l(u) = \frac{1 + \sqrt{3}}{\sqrt{2}}\sinh u - (\sqrt{3} - 1)\sin v$. But $\frac{dl}{du} = \frac{1 + \sqrt{3}}{\sqrt{2}}\cosh u - \frac{(\sqrt{3} - 1)^2}{\sqrt{2}}\cos v > \frac{1 + \sqrt{3}}{\sqrt{2}} - \frac{(\sqrt{3} - 1)^2}{\sqrt{2}} = \frac{3\sqrt{3} - 3}{\sqrt{2}} > 0$ hence l increases from $l(0) = 0$ for $u > 0$, therefore $l > 0$.

Then $\frac{dh}{du} > 0$ for any $u > 0$ and, because $h(0) = 0$, it follows that $h(u) > 0$ for $u > 0$. For $u \in (0, u^*)$ where $u^*\frac{\sqrt{3} - 1}{\sqrt{2}} = v^* = \pi$ (and consequently, $u^* \approx 6$) we have $(u - \sqrt{2}\, v)\sin v = (2 - \sqrt{3})u \sin v > 0$ and (A3.2) has no solution becuse the left-hand side $(-h(u))$ is negative while the right-hand side is positive. On the other hand for $u > 2$ the function $g(u) = \frac{3+\sqrt{3}}{\sqrt{2}} u \sinh u - \sqrt{2} \cosh u - \sqrt{2} - (2\sqrt{3} - 3)u$ is strictly positive because $\frac{dg}{du} = \frac{1+\sqrt{3}}{\sqrt{2}} \sinh u + \frac{3+\sqrt{3}}{\sqrt{2}} u \cosh u + 3 - 2\sqrt{3} > 0$ and $g(2) > 0$. Since (A3.2) expresses, equivalently, in the form

$$f(u) \equiv \frac{3+\sqrt{3}}{\sqrt{2}} u \sinh u - \sqrt{2}\left[\cosh u - \cos\left(\frac{\sqrt{3}-1}{\sqrt{2}} u\right)\right] + \sqrt{3}(2 - \sqrt{3})u \sin\left(\frac{\sqrt{3}-1}{\sqrt{2}} u\right) = 0$$

and $f(u) > g(u) > 0$ if follows that (A3.2) has no solutin for $u > 2$. Consequently (A3.1) has no solution for $u > 0$.

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