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DISCRETE PIECEWISE LINEAR l_1 APPROXIMATION

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Abstract. In this paper the piecewise linear function in many variables is obtained by the discrete l_1 approximation using the linear or the linear mixed integer programming. The censored discrete linear l_1 approximation can be extended using given results.

1. Introduction. Consider a sample of statistical data for n random variables which must not be independent, and one dependent random variable. Our aim is to obtain a piecewise linear function for which the sum of absolute deviations is minimal. The piecewise linear function is important in the optimization of the business process [2]. For i-th element of the sample the deviation is defined as the difference between the observed value of the dependent variable y_i and the function value $f(x_i)$. Here $x_i \in R_n$ is a vector whose components are observed values of n random variables in the i-th element of the sample and $f: R^n \to R$ is a piecewise linear function. This problem can be expressed in the form

$$\text{minimize} \sum_{i=1}^{m} |y_i - f(x_i)| \tag{1.1}$$

The function f contains unknown parameters which must be estimated. The number of observations m must be greater than the number of unknown parameters.

The absolute value function can be expressed using zero-one variables subject to additional constraints [2]. The function

$$g(x) = |y - f(x)|$$

can be replaced in the programming problem by

$$g(x) = r + s$$

subject to nonnegative variables r and s, suitable constant c and

$$y - f(x) = r - s \tag{1.2}$$

$$r \leqslant cu \tag{1.3}$$

$$s \leqslant c(1 - u) \tag{1.4}$$

$$u = 0 \text{ or } 1. \tag{1.5}$$

If the absolute value function arises in the objective function only and this function is minimized, (1.3) — (1.5) can be omitted. Therefore instead of (1.1) we obtain

$$\text{minimize } \sum_{i=1}^{m} (r_i + s_i) \tag{1.6}$$

subject to nonnegative variables r_i and s_i and

$$y_i - f(x_i) = r_i - s_i \qquad i = 1, \ldots, m \tag{1.7}$$

The piecewise linear function of few variables can be expressed in different forms. We will take the form [1]

$$f(x) = \max(g_1(x),\ g_2(x), \ldots,\ g_h(x)) \tag{1.8}$$

which can be used also in case where many variables arise. If g_i are linear functions, $f(x)$ is a convex piecewise linear function. Similarly the function.

$$f(x) = \min\ (g_1(x),\ g_2(x), \ldots,\ g_h(x))$$

is a concave piecewise linear function if g_i are linear functions. The general piecewise linear function can be expressed in the form (1.8), where

$$g_i(x) = \min(g_{i1}(x),\ g_{i2}(x), \ldots, g_{ik_i}(x)) \quad i = 1, \ldots, h \tag{1.9}$$

and $g_{ij} : R^n \to R$ are linear functions.

2. Estimation of Parameters by the Mixed Integer Model. THEOREM. *If g_1 and g_2 are bounded functions, then the function*

$$f(x) = \max\ (g_1(x),\ g_2(x))$$

can be replaced by

$$f(x) = g_2(x) + q \tag{2.1}$$

subject to (1.5), *nonnegative variables p and q and additional constraints*

$$g_1(x) - g_2(x) = q - p \tag{2.2}$$

$$p \leqslant cu \tag{2.3}$$

$$q \leqslant c(1 - u) \tag{2.4}$$

where c is a suitable positive constant.

Proof. If

$$g_1(x) - g_2(x) > 0 \tag{2.5}$$

then from (2.2) it follows $q > 0$, since p is nonnegative. Therefore from (2.4) and (1.5) it follows $u = 0$. From (2.3) therefore it follows $p = 0$. Since g_1 and g_2 are bounded functions and c is a chosen constant, from (2.2) it follows

$$g_1(x) - g_2(x) = q$$

Then from (2.1) it follows

$$f(x) = g_1(x)$$

and the theorem is proved subject to (2.5). Similarly in case

$$g_1(x) - g_2(x) < 0$$

it follows $u = 1$, $q = 0$ and

$$f(x) = g_2(x)$$

and the theorem is proved also in this case. If

$$g_1(x) = g_2(x),$$

from (2.2) — (2.4) and (1.5) it follows $p = q = 0$, since p and q are nonnegative. Therefore the theorem holds also in this case. So the theorem is proved.

In similar way the function

$$f(x) = \min\ (g_1(x),\ g_2(x))$$

can be replaced. If

$$g_i(x) = \max\ (g_{i1}(x),\ g_{i2}(x))$$

or

$$g_i(x) = \min\ (g_{i1}(x),\ g_{i2}(x))$$

for $i = 1,2$, then first $g_{i1}(x)$ and $g_{i2}(x)$ must be transformed. Using this result the function (1.8) subject to (1.9) can be transformed.

Consider the function

$$f(x) = \max\ (\max\ (g_1(x), g_2(x)),\ \max\ (g_3(x),\ g_4(x))). \tag{2.6}$$

Using (2.1) — (2.4) we obtain

$$\max\ (g_1(x),\ g_2(x)) = g_2(x) + q_1 \tag{2.7}$$

$$g_1(x) - g_2(x) = q_1 - p_1 \tag{2.8}$$

$$p_1 \leqslant cu \tag{2.9}$$

(2.10) $$q_1 \leqslant c(1-u)$$

(2.11) $$\max\ (g_3(x),\ g_4(x)) = g_4(x) + q_2$$

(2.12) $$g_3(x) - g_4(x) = q_2 - p_2$$

(2.13) $$p_2 \leqslant cv$$

(2.14) $$q_2 \leqslant c(1-v)$$

where p_i and q_i are nonnegative, u and v are zero-one variables and c is a suitable constant. Considering (2.7), (2.11), (2.6) and the given theorem we obtain

(2.15) $$f(x) = g_4(x) + q_2 + q_3$$

subject to nonnegative variables p_i and q_i, zero-one variables u, v and w, (2.8) — (2.10), (1.12) — (2.14) and

(2.16) $$g_2(x) + q_1 - g_4(x) - q_2 = q_3 - p_3$$

(2.17) $$p_3 \leqslant cw$$

(2.18) $$q_3 \leqslant c(1-w)$$

Consider the problem (1.1) subject to (2.6). Using (1.6), (1.7) and this result in can be written in the form

(2.19) $$\text{minimize } \sum_{i=1}^{m} (r_i + s_i)$$

subject to nonnegative variables p_i, q_i, r_i and s_i, zero-one variables u_i, v_i, and w_i, suitable constant c and

(2.20) $$g_4(x_i) + q_{i2} + q_{i3} + r_i - s_i = y_i$$

(2.21) $$g_1(x_i) - g_2(x_i) = q_{i1} - p_{i1}$$

(2.22) $$p_{i1} \leqslant cu_i$$

(2.23) $$q_{i1} \leqslant c(1-u_i)$$

(2.24) $$g_3(x_i) - g_4(x_i) = q_{i2} - p_{i2}$$

(2.25) $$p_{i2} \leqslant cv_i$$

(2.26) $$q_{i2} \leqslant c(1-v_i)$$

(2.27) $$g_2(x_i) + q_{i1} - g_4(x_i) - q_{i2} = q_{i3} - p_{i3}$$

(2.28) $$p_{i3} \leqslant cw_i$$

(2.29) $$q_{i3} \leqslant c(1-w_i)$$

for $i = 1,\ldots,m$. If $g_1,\ldots,g_4$ are linear functions, then (2.19) — (2.29) is a linear mixed integer programming problem with zero-one variables u_i, v_i and w_i, nonnegative variables q_{ij}, p_{ij}, r_{ij} and s_{ij} and unbounded parameters of linear functions.

3. Estimation of Parameters by the Linear Model. The linear mixed integer programming problem (2.19) — (2.29) can have a lot of variables and constraints. If the sample has 100 elements and the function has four linear pieces in the programming problem arise 300 zero-one variables, 800 nonnegative variables and 1000 constraints. In spite of the development of the linear mixed integer programming and the corresponding computational technics [4] such a programming problem can cause difficulties. Therefore it is reasonable to approximate this model by a model, which can be simply solved.

The problem (2.19) subject to (2.20) — (2.29) can be approximated by the linear programming problem. Consider the linear function $h: R_n \to R$. If

(3.1) $$d_{k-1} \leqslant h(x_i) < d_k$$

where constants d_k satisfy the condition

$$d_{k-1} < d_k \text{ for } k = 1,\ldots,4$$

then

(3.2) $$\max\ (\max\ (g_1(x_i),\ g_2(x_i)),\ \max\ (g_3(x_i),\ g_4(x_i))) = g_k(x_i)$$

Constants d_k can be determined so that for any element of the sample for which x_i satisfies the condition (3.1) an index k exists, and parameters of linear functions g_k can be determined so that (3.2) is true.

If x_i satisfies (3.1) for $k = 1$, then from (3.2) it follows

$$g_1(x_i) \geqslant g_2(x_i)$$

Therefore from (2.7) — (2.10) and $q_{i1} \geqslant 0$ it follows $p_{i1} = 0$ and we can take $u_i = 0$. Since

$$\max\ (g_1(x_i),\ g_2(x_i)) \geqslant \max\ (g_3(x_i),\ g_4(x_i)),$$

from (2.7), (2.11), (2.16) — (2.18) and $q_{i3} \geqslant 0$ it follows $p_{i3} = 0$ and we can take $w_i = 0$. Since (2.6) is a convex function the zero-one variable v_i can be determined as well. If there exists at least one element of the sample, which satisfies the condition (3.1) for $k = 4$, then any element, which satisfies the condition (3.1) for $k < 4$, satisfies the condition

$$g_3(x_i) \geqslant g_4(x_i).$$

Therefore we obtain $p_{i2} = 0$ and we can take $v_i = 0$. Instead of (2.21) — (2.29) in our case we obtain

(3.3) $$g_1(x_i) - g_2(x_i) = q_{i1}$$
$$g_3(x_i) - g_4(x_i) = q_{i2}$$

(3.4) $$g_2(x_i) + q_{i1} - g_4(x_i) - q_{i2} = q_{i3}.$$

Using (3.4) and (3.3) from (2.20) it follows

$$g_1(x_i) + r_i - s_i = y_i$$

If x_i satisfies (3.1) for $k = 2$, then $u_i = 1$ and $v_i = w_i = 0$. In this case we obtain following constraints

$$g_2(x_i) + r_i - s_i = y_i$$
$$g_1(x_i) - g_2(x_i) \leqslant 0$$
$$g_3(x_i) - g_4(x_i) = q_{i2}$$
$$g_2(x_i) - g_4(x_i) - q_{i2} = q_{i3}$$

If x_i satisfies (3.1) for $k = 3$, then we obtain

$$g_3(x_i) + r_i - s_i = y_i$$
$$g_1(x_i) - g_2(x_i) \leqslant 0$$
$$g_3(x_i) - g_4(x_i) = q_{i2}$$
$$g_2(x_i) - g_4(x_i) - q_{i2} \leqslant 0$$

If x_i satisfies (3.1) for $k = 4$, we obtain

$$g_4(x_i) + r_i - s_i = y_i$$
$$g_1(x_i) - g_2(x_i) \leqslant 0$$
$$g_3(x_i) - g_4(x_i) \leqslant 0$$
$$g_2(x_i) - g_4(x_i) \leqslant 0$$

For this problem a matrix generator can be useful.

The function (2.6) can be extended to more linear pieces and to concave piecewise linear functions. Since the convexity is used for the determination of zero-one variables, the extension of this method to functions (1.8) subject to (1.9) can not be used. But the determination of zero-one variables is simple also in this case. If

$$f(x) = \max(\min(g_1(x), g_2(x)), \min(g_3(x), g_4(x))),$$

instead of (3.1) we can take

$$h_1(x_i) \leqslant d_1 \tag{3.5}$$

$$h_2(x_i) \leqslant d_2 \tag{3.6}$$

where h_1 and h_2 are different linear functions. If x_i satisfies the condition (3.5), then we can take

$$\min(g_1(x_i), g_2(x_i)) = g_1(x_i)$$

and

$$\min(g_3(x_i), g_4(x_i)) = g_3(x_i).$$

If x_i does not satisfy (3.5), then we take

$$\min(g(x_i), g_2(x_i)) = g_2(x_i)$$
$$\min(g_3(x_i), g_4(x_i)) = g_4(x_i)$$

Similarly the condition (3.6) can be used for the determination of the maximum.

4. **Censored linear l_1 approximation;** Consider the problem [3], [5]

$$\text{minimize} \sum_{i=1}^{m} |y_i - \max(z_i, a' x_i)| \tag{4.1}$$

where $x_i \in R^n$, $y_i \in R$ and $z_i \in R$ are observed and $a \in R^n$ must be estimated. Using (1.6), (1.7) and (2.1) — (2.4), this problem can be written in the form

$$\text{minimize} \sum_{i=1}^{n} (r_i + s_i)$$

subject to nonnegative variables r_i, s_i, p_i and q_i, zero-one variables u_i and

$$y_i - a' x_i - q_i = r_i - s_i$$
$$z_i - a' x_i = q_i - p_i$$
$$p_i \leqslant c u_i$$
$$q_i \leqslant c(1 - u_i)$$

for $i = 1, \ldots, m$. Now we have a mixed integer programming problem with $4m$ constraints, $4m$ nonnegative variables, m zero-one variables and n unbounded components of the vector a.

This result can be useful although an algorithm for this problem exists [5]. Since computer programs for the linear mixed integer programming are available we can use them. Therefore special computer programs are not needed.

Using given results problem (4.1) can be extended if we take

$$\text{minimize} \sum_{i=1}^{m} |y_i - \max(z_i, \min(a' x_i, t_i))|$$

where $t_i \in R$ is observed as well. This problem can be used in statistics in a similar way as problem (4.1). Instead of the linear function $a'x$ we can take the piecewise linear function.

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