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GENERAL ESTIMATES FOR THE LINEAR POSITIVE OPERATORS WHICH PRESERVE LINEAR FUNCTIONS

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Let I be, an arbitrary interval of the real axis, $J \subset I$ be a subinterval and $\mathscr{F}(I)$ respectively $\mathscr{F}(J)$ the spaces of the real-valued functions defined on I and respectively on J. We denote by $\mathscr{F}_b(I)$ the subspace of $\mathscr{F}(I)$ of those functions that are bounded on each compact (subinterval of I. Let $U \subset \mathscr{F}_b(I)$ be a subspace and $L: U \to \mathscr{F}(J)$ be a linear positive operator that preserves linear functions. In the present paper we give a new method to obtain estimates involving the second order modulus of continuity for the difference $L(f, x) - f(x)$ when $f \in U$ and $x \in J$. These estimates are in some instances better and more general than the estimates that we have obtained otherwise in [4]. By combining these two methods there results an estimate that improves the earlier ones (see [3]).

Since for a fixed point $x \in I$, $L(f, x)$ is a linear positive functional, in the first section we shall express the estimates in terms of functionals, and in the second we shall apply these results to the operators of S. N. Bernstein.

1. Estimates for linear positive functionals. In this section x_0 will be a fixed point of the arbitrary interval I. For any $f \in \mathscr{F}(I)$ we denote by $\delta^+ f$ and $\delta^- f$ the functions of $\mathscr{F}(I)$ that are defined in the following mode:

(1) $$(\delta^+ f)(t) = \begin{cases} f(t) - f(x_0), & t \in I,\ t > x_0 \\ 0, & t \in I,\ t \leqslant x_0 \end{cases}$$

$$(\delta^- f)(t) = \begin{cases} f(t) - f(x_0), & t \in I,\ t < x_0 \\ 0, & t \in I,\ t \geqslant x_0 \end{cases}$$

From (1) there results the following representation:

(2) $$f(t) = (\delta^+ f)(t) + (\delta^- f)(t) + f(x_0), \text{ for } t \in I,\ f \in \mathscr{F}(I).$$

We denote by $e_i \in \mathscr{F}(I)$ the functions defined by $e_i(t) = t^i$ for $t \in I$. Also, for any real number $a \in \mathbb{R}$ we denote by $]a[$ the greatest integer that is less than a, and by $[a]$ the greatest integer that is less than a or equal to a.

We give in the following definition the general environment in which we shall work.

Definition 1. *Let V be a linear subspace of $\mathscr{F}_b(I)$ such that*

(a) $e_i \in V$, $i = 0, 1, 2$

(b) If $f \in V$ then $|f| \in V$.

(c) If $f \in V$ and $g \in \mathscr{F}_b(I)$ are such that $|g| \leqslant |f|$ then $g \in V$.

Let $F : V \to \mathbb{R}$ be a linear positive functional with the property: $F(e_i) = x_0^i$, $i = 0,1$.

A particular but sufficiently general for application, case of Definition 1, is contained in the definition from below.

Definition 2. Let μ be a positive Borel measure on I with the property: $\int_I e_i \, d\mu = x_0^i$ for $i = 0,1$ and $\int_I e_2 \, d\mu < \infty$. We denote by $V = \mu\mathscr{L}(I)$ the space of real-valued functions defined on I that are μ-integrable. Let the linear positive functional $F: V \to \mathbb{R}$ be defined by: $F(f) = \int_I f \, d\mu$, $f \in V$.

We shall give some lemmas.

Lemma 1. *For a functional F as in Definition 1 we have:*

$$F(|\delta^+ e_1|) = F(|\delta^- e_1|). \tag{3}$$

Proof. From (2) there follows $e_1 - x_0 e_0 = \delta^+ e_1 + \delta^- e_1$, and by the definition of F there follows $0 = F(e_1 - x_0 e_0) = F(\delta^+ e_1) + F(\delta^- e_1)$. Then $F(|\delta^+ e_1|) = F(\delta^+ e_1) = -F(\delta^- e_1) = F(-\delta^- e_1) = F(|\delta^- e_1|)$

Remark 1. For a functional as in Definition 2, Lemma 1 can be expressed by the following equality:

$$\int_{I^-} |e_1 - x_0 e_0| \, d\mu = \int_{I^+} |e_1 - x_0 e_0| \, d\mu, \tag{3'}$$

where $I^- = I \cap (-\infty, x_0)$ and $I^+ = I \cap (x_0, \infty)$.

We shall write $M_F = F(|\delta^+ e_1|)$. Also if $g(t_1, \ldots, t_n)$ is a function of several variables we shall denote by $F_{t_i}(g(t_1, \ldots, t_n))$ the value of the functional F applied to the partial function $t_i \to g(t_1, \ldots, t_n)$ with t_j=const. $(i \neq j)$.

Lemma 2. *For a functional F as in Definition 1, if $M_F \neq 0$ then*

(4) $F(f) - f(x_0) = F_{t_1}(F_{t_2}(\varphi_f(t_1, t_2)))$ *for any* $f \in V$,

where $\varphi_f : I \times I \to \mathbb{R}$ is defined by:

$$\varphi_f(t_1, t_2) = (M_F)^{-1}\{(|\delta^+ e_1|(t_2))\,((\delta^- f)(t_1)) + (|\delta^- e_1|(t_1))\,((\delta^+ f)(t_2))\}$$

for $(t_1, t_2) \in I^2$.

Proof. Since $|\delta^+ e_1|(t_2) = (\delta^+ e_1)(t_2)$ and $|\delta^- e_1|(t_1) = -(\delta^- e_1)(t_1)$, it follows that the function $t_2 \to \varphi_f(t_1, t_2)$ belongs to V for any $t_1 \in V$.

Next, the function $t_1 \to F_{t_2}(\varphi_f(t_1, t_2))$ belongs to V since $F_{t_2}\,\varphi_f(t_1, t_2)) = (\delta^- f)(t_1) + (M_F)^{-1} F((\delta^+ f))\,(|\delta^- e_1|(t_1))$. Finally: $F_{t_1}(F_{t_2}(\varphi_f(t_1, t_2))) = F(\delta^- f) + F(\delta^+ f) = F(f - f(x_0)e_0) = F(f) - f(x_0)$.

Remark 2. If F is a functional as in Definition 2 the assertion of Lemma 2 can be expressed by the following equality:

$$F(f) - f(x_0) = \int_{I^- \times I^+} \varphi_f(t_1, t_2) \, d\mu(t_1) \times \mu(t_2), \tag{4'}$$

where $\mu \times \mu$ is the product measure of μ with itself on $I^- \times I^+$. Indeed, since the functions $|\delta^\pm e_1|$ and $\delta^\pm f$ are μ-measurable it results that φ_f is $\mu \times \mu$-measurable. Then we have:

$$(M_F)^{-1} \int_{I^-} \left\{ \int_{I^+} (|\delta^+ e_1|(t_2))\,(|\delta^- f|(t_1)) \, d\mu(t_2) \right\} d\mu(t_1) +$$

$$+ (M_F)^{-1} \int_{I^-} \left\{ \int_{I^+} (|\delta^- e_1|(t_1))(|\delta^+ f|(t_2)) \, d\mu(t_1) \right\} d\mu(t_2) \leqslant$$

$$\leqslant \int_I |f| \, d\mu + |f(x_0)| < \infty.$$

Hence φ_f is $\mu \times \mu$-integrable and assertion (4') can be obtained by applying Fubini's theorem.

Lemma 3. *For a functional F as in Definition 2 the following assertions are equivalent:*

(*a*) $M_F = 0$

(*b*) $F(f) = f(x_0)$ *for any* $f \in V$.

Proof. One can infer b) $\to$ a) by choosing $g \in V$ defined by: $g(t) = 1 \ (t \in I^-)$ and $g(t) = 0 \ (t \in \{x_0\} \cup I^+)$. Then $\mu(I^-) = 0$. The inference a) $\to$ b) follows from the fact that for any $n \geqslant 1$, $0 = M_F = \int_{I^+} |e_1 - x_0 e_0| \, d\mu \geqslant (1/n)\,\mu(I \cap (x_0 + (1/n), \infty))$, and consequently $\mu(I^+) = 0$. Analogously there results $\mu(I^-) = 0$ and on the other hand $\mu(I) = 1$ and then $\mu(\{x_0\}) = 1$.

Remark 3. One cannot obtain an analogous lemma for the general functionals as in Definition 1.

Definition 3. (a) For every $f \in F(I)$ and every points of I: $t_1 < x < t_2$ we write:

$$\Delta(f, t_1, t_2, x) = \frac{t_2 - x}{t_2 - t_1} f(t_1) + \frac{x - t_1}{t_2 - t_1} f(t_2) - f(x). \tag{5}$$

(b) For every $f \in \mathscr{F}(I)$ and every real number $\rho > 0$ we write:

$$\omega_1(f, \rho) = \sup \{ |f(x+h) - f(x)|, \ x, x+h \in I, \ 0 < h \leqslant \rho \}$$

$$\omega_2(f, \rho) = \sup \{ |f(x+h) - 2f(x) + f(x-h)|, \ x-h, \ x+h \in I, \ 0 < h \leqslant \rho \}. \tag{6}$$

$$\omega_2^*(f, \rho) = \sup \{ |\Delta(f, t_1, t_2, x)|, \ t_1, t_2, t_2 - t_1 \leqslant \rho \}.$$

Remark 4. We admit the possibility that $\omega_1(f, \rho)$, $\omega_2(f, \rho)$ and $\omega_2^*(f, \rho)$ are infinite.

LEMMA 4. *If we consider a fixed function $f \in F_b(I)$ we have:*

(a) *The function $\rho \to \omega_2^*(f, \rho)$ is nondecreasing on $(0, \infty)$.*

(b) *For any real number $\rho > 0$ we have:*

$$\omega_2^*(f, \rho) \leqslant \omega_1(f, \rho) \ \textit{and}$$
$$(1/2)\, \omega_2(f, \rho/2) \leqslant \omega_2^*(f, \rho) \leqslant \omega_2(f, \rho/2) \tag{7}$$

(c) *If f is uniformly continuous function then $\omega_*^2(f, \rho) \to 0$ $(\rho \to 0+)$ and conversely, if this limit exists then the function f is continuous.*

(d) *If t_1, x, t_2 belong to I and $t_1 < x < t_2$ then for any $\rho > 0$.:*

$$|\Delta(f, t_1, t_2, x)| \leqslant (t_2 - t_1)^{-1} \{ (t_2 - x)(1 +](x - t_1)/\rho[)^2 + (x - t_1)(1 +](t_2 - x)/\rho[)^2 \} \omega_2^*(f, 2\rho) \tag{8}$$

Proof. (a) It is obvious from Definition 3.

(b) Let $t_1 < x < t_2$, $t_1, t_2 \in I$ be such that $t_2 - t_1 \leqslant \rho$. The inequality $\omega_2^*(f, \rho) \leqslant \omega_1(f, \rho)$ follows from the inequality:

$$|\Delta(f, t_1, t_2, x)| \leqslant \frac{t_2 - x}{t_2 - t_1} |f(t_1) - f(x)| + \frac{x - t_1}{t_2 - t_1} |f(t_2) - f(x)|.$$

If $x - h$, $x + h \in I$ are such that $0 < h \leqslant \rho/2$ then we have: $f(x+h) - 2f(x) + f(x-h) = 2\Delta(f, x-h, x+h, x)$. Therefore $(1/2)\, \omega_2(f, \rho/2) \leqslant \omega_2^*(f, \rho)$.

Now, let $t_1 < t_2$, $t_1, t_2 \in I$ be such that $t_2 - t_1 \leqslant \rho$ and let $x \in (t_1, t_2)$ be a variable point. Let us consider the polynomial $p(x) = -f(t_1) + \frac{f(t_2) - f(t_1)}{t_2 - t_1}(t_1 - x)$, and we put $g(x) = f(x) + p(x)$. We have $g(t_1) = 0 = g(t_2)$ and since p is a polynomial of degree one there follows: $\Delta(f, t_1, t_2, x) = \Delta(g, t_1, t_2, x) = -g(x)$ and $\omega_2(f, \rho/2) = \omega_2(g, \rho/2)$. Since f is bounded on $[t_1, t_2]$, let $M = \sup \{ |g(t)|, t \in [t_1, t_2] \}$. We have only to consider the case where $M > 0$.

For $0 < \varepsilon < M/2$ arbitrarily chosen there is $u \in (t_1, t_2)$ such that $|g(u)| > M - \varepsilon$. We shall consider only the case where $g(u) > 0$ and

$u \geqslant (t_1 + t_2)/2$, since the others are reductible to this. We have $g(t_2) - 2g(u) + g(2u - t_2) \leqslant -g(u) + \varepsilon$. Consequently $\omega_2(g, \rho/2) \geqslant g(u) - \varepsilon \geqslant M - 2\varepsilon \geqslant |g(x)| - 2\varepsilon = |\Delta(g, t_1, t_2, x)| - 2\varepsilon = |\Delta(f, t_1, t_2, x)| - 2\varepsilon$. Therefore $\omega_2(f, \rho/2) \geqslant \omega_2^*(f, \rho) - 2\varepsilon$ for any $M/2 > \varepsilon > 0$. Thus (b) is completely proved.

(c) If f is uniformly continuous, we have $\omega_1(f, \rho) \to 0$ $(\rho \to 0+)$ and then from the inequality proved at point b) it follows that $\omega_2^*(f, \rho) \to 0 (\rho \to 0+)$.

Conversely, let us suppose that f is not continuous at the point $u^* \in I$. Let us consider for instance that f is not continuous from the right at u^*. Then there is a nonincreasing sequence (u_n) with the limit u^* and $f(u_n) \to q$ $(n \to \infty)$, $q \neq f(u^*)$. Let $\rho > 0$ fixed. We choose $u_{n_0} \in (u^*, u^* + \rho)$. We have $\omega_2^*(f, \rho) \geqslant |\Delta(f, u^*, u_{n_0}, u_{n_0+p})|$ for any $p \in \mathbb{N}$. But $\lim_{p \to \infty} |\Delta(f, u^*, u_{n_0}, u_{n_0+p})| = |f(u^*) - q|$. Hence $\omega_2^*(f, \rho) \geqslant f(u^*) - q|$ for any $\rho > 0$. We have obtained a contradiction.

(d) In order to simplify the notation we put:

$$B(y, t, u) = \Delta(f, y - t, y + u, y) = \frac{u}{t+u} f(y - t) + \frac{t}{t+u} f(y + u) - f(y) \tag{9}$$

for $y - t$, $y + u \in I$, $t, u > 0$.

For $0 < h < t$ the following identity:

$$\frac{u}{t+u} f(y-t) + \frac{t}{t+u} f(y+u) - f(y) = \frac{t(u+h)}{h(t+u)} \left\{ \frac{u}{u+h} f(y-h) + \frac{h}{u+h} f(y+u) - f(y) \right\} + \frac{ut}{h(t+u)} \left\{ \frac{h}{t} f(y-t) + \frac{t-h}{t} f(y) - f(y-h) \right\},$$

proves the equality:

$$B(y, t, u) = \frac{t(u+h)}{h(t+u)} B(y, h, u) + \frac{ut}{h(t+u)} B(y-h, t-h, h) \tag{10}$$

for $0 < h < t$.

If we take in (10) $t = mh$ and $u = h$, where $h > 0$ and $m \in \mathbb{N}$, $m \geqslant 2$ we obtain:

$$B(y, mh, h) = 2\frac{m}{m+1} B(y, h, h) + \frac{m}{m+1} B(y - h, (m-1)h, h). \tag{11}$$

The following equality :

$$(12)\qquad B(y,\,mh,\,h)=\frac{2}{m+1}\sum_{j=0}^{2}(m-j)\,B(y-jh,\,h,\,h)+$$

$$+\frac{m-k}{m+1}\,B(y-(k+1)h,\,(m-k-1)h,\,h),$$

$$\text{for } 0\leqslant k\leqslant m-2,\ h>0,\ m\geqslant 2$$

can be proved by induction over k. Indeed, for $k=0$ the equality in (12) is equivalent to (11). Afterwards if $0\leqslant k<m-2$ the induction step $k\rightarrow k+1$ results from the following equality :

$$\frac{m-k}{m+1}\,B(y-(k+1)h,\,(m-k-1)\,h,\,h)=$$

$$=2\,\frac{m-k-1}{m+1}\,B(y-(k+1)\,h,\,h,\,h)+$$

$$+\frac{m-k-1}{m+1}\,B(y-(k+2)h,\,(m-k-2)h,\,h),$$

that is a consequence of (10).

From assertion (12) for $k=m-2$ we obtain :

$$|B(y,\,m\,h,h)\,|\leqslant\left(\frac{2}{m+1}\sum_{j=0}^{m-2}(m-j)+\frac{2}{m+1}\right)\omega_2^*(f,2h)=m\omega_2^*(f,2h).$$

Consequently the following inequality is true :

$$(13)\qquad |\,B(\,y,\,mh,\,h)\,|\leqslant m\omega_2^*(f,\,2\rho),\ \text{if } 0<h\leqslant\rho \text{ and } m\geqslant 1\,.$$

Now, let $t>0$ and $0<u\leqslant\rho$ be such that $y-t,\ y+u\in I$. If $t\leqslant\rho$ then $|B(y,\,t,\,u)|\leqslant\omega_2^*(f,\,2\rho)$. If $t>\rho$ let us write $m=\,]t/\rho[$ and $h=\dfrac{t}{m+1}$. Hence $m\geqslant 1$ and $h\leqslant\rho$. By applying (10) and (13) we have :

$$|B(y,\,t,\,u)\,|\leqslant\frac{t\,(u+h)}{h(t+u)}\,|\,B(y,\,h,\,u)\,|+\frac{ut}{h(t+u)}\,|B(y-h,t-h,h)|\leqslant$$

$$\leqslant\frac{t(u+h)+utm}{h(t+u)}\,\omega_2^*(f,\,2\rho)=\frac{t+u\,(1+m)^2}{t+u}\,\omega_2^*\,(f,\,2\rho).$$

Therefore in both the cases $t\leqslant\rho$ and $t>\rho$ we have :

$$(14)\qquad |\,B(y,\,t,\,u)\,|\leqslant\left\{1+\frac{u}{t+u}\,(2]\,t/\rho[\,+\,(]t/\rho\,[)^2\right\}\,\omega_2^*(f,\,2\rho),$$

if $0<t$ and $0<u\leqslant\rho$ with $y-t,\ y-u\in I$.

The symmetry of (14) proves the following inequality :

$$(15)\qquad |B(y,\,t,\,u)|\leqslant\left\{1+\frac{t}{t+u}\,(2]\,u/\rho\,[\,+\,(]u/\rho[)^2)\right\}\,\omega_2^*(f,\,2\,\rho),$$

if $0<t\leqslant\rho$ and $0<u$ with $y-t,\ y+u\in I$.

Finally let us consider the case where $t>\rho$ and $u>\rho$. Let $m=\,]t/\rho[$ and $h=t/(m+1)$. By using (10), (13) and (15) there follows

$$|\,B(y,\,t,\,u)\,|\leqslant\frac{t(u+h)}{h(t+u)}\,|\,B(y,\,h,\,u)\,|+\frac{ut}{h(t+u)}\,|\,B(y-h,t-h,h)\,|\leqslant$$

$$\leqslant\frac{t\,(\,u+h)}{h(t+u)}\left\{1+\frac{h}{u+h}\,(2]u/\rho[+(]u/\rho[)^2)\right\}\omega_2^*(f,\,2\,\rho)+$$

$$+\frac{u\,tm}{h(t+u)}\,\omega_2^*(f,\,2\,\rho)=\left\{\frac{t+u(1+m)^2}{t+u}+\right.$$

$$\left.+\frac{t}{t+u}\,(2]u/\rho[\,+\,(]u/\rho[)^2)\right\}\,\omega_2^*\,(f,\,2\,\rho).$$

From the last inequality, by taking into account (14) and (15) we infer that the following inequality

$$(16)\qquad |B(y,\,t,\,u)\,|\leqslant(t+u)^{-1}\,\{u(1+\,]t/\rho[)^2+t(1+\,]u/\rho\,[)^2\,\omega_2^*\,(f,\,2\,\rho)$$

holds in the general case where $y-t,\ y+u\in I,\ u,t>0,\ \rho>0$. Then (8) follows from (16) by taking $y=x,\ t=x-t_1$ and $u=t_2-x$.

Our main result is the following theorem.

THEOREM 1. *Let F be a functional as in Definition 1 such that $M_F\neq 0$. Then :*

$$(17)\qquad |F(f)-f(x_0)\,|\,\leqslant F(\theta_\rho)\,\omega_2^*\,(f,\,2\,\rho),$$

for any $f\in V$ and any real number $\rho>0$, where by θ_ρ we denote the function : $\theta_\rho=(1+]\,|\,e_1-x_0e_0\,|\,/\,\rho[)^2$.

Proof. By Definition 1 it results that $\theta_\rho\in V$. From Lemma 2 we have : $|F(f)-f(x_0)\,|=|F_{t_1}\,(F_{t_2}\,(\varphi_f(t_1,\,t_2)))\,|\leqslant F_{t_1}(F_{t_2}\,(\,|\,\varphi_f(t_1,\,t_2)\,|))$.

Since :

$$\varphi_f\,(t_1,\,t_2)=\begin{cases}\dfrac{t_2-t_1}{M_F}\,\Delta(f,\,t_1,\,t_2,\,x_0) & \text{if } t_1<x_0<t_2\\ 0 & \text{if } t_1\geqslant x_0 \text{ or } t_2\leqslant x_0,\end{cases}$$

it results from (8) that :

$$|\,\varphi_f(t_1,t_2)\,|\leqslant\{(1/M_F)\,(\,|\,\delta^+e_1\,|\,(t_2))\,(1+\,]\,|\,\delta^-e_1\,|\,(t_1)/\rho\,[)^2+$$

$$+\,(1/M_F)\,(\,|\delta^-e_1\,|\,(t_1))\,(1+\,]\,|\,\delta^+e_1\,|\,(t_2)/\rho\,[)^2\,\omega_2^*\,(f,\,2\,\rho).$$

Consequently:

$$F_{t_1}(F_{t_2}(|\varphi_f(t_1,t_2)|)) \leqslant F((1 +]|\delta^- e_1|/\rho[)^2 + $$
$$+(1 +]|\delta^+ e_1|/\rho[)^2)\, \omega_2^*(f, 2\rho)$$

The theorem is then proved if we take into account the equality:

$$(1 +]|\delta^- e_1|/\rho[)^2 + (1 +]|\delta^+ e_1|/\rho[)^2 =$$
$$= (1 +]|e_1 - x_0 e_0|/\rho[)^2.$$

COROLLARY 1. *Let F be a functional as in Definition 1 such that $M_F \neq 0$ or as in Definition 2. Then we have:*

i) $|F(f) - f(x_0)| \leqslant F(\theta_\rho)\omega_f(\rho)$

ii) $|F(f) - f(x_0)| \leqslant$

(18) $\leqslant \{1 + 2F(]|e_1 - x_0 e_0|/\rho[) + \rho^{-2}F(\psi)\}\, \omega_f(\rho)$

iii) $|F(f) - f(x_0)| \leqslant \{1 + 3\rho^{-2}F(\psi)\}\,\omega_f(\rho)$,

for any $f \in V$ and any real number $\rho > 0$, where $\omega_f(\rho)$ is anyone of the moduli of continuity $\omega_2^*(f, 2\rho)$, $\omega_1(f, 2\rho)$ or $\omega_2(f, \rho)$ and ψ is the function $\psi = (e_1 - x_0 e_0)^2$.

Proof. Corollary results from the inequality $(]|e_1 - x_0 e_0|/\rho[)^j \leqslant \rho^{-2}\psi$ for $j = 1, 2$, from Lemma 4 — (7), Lemma 3 and Theorem 1.

Remark 5. For the modulus of continuity of the second order we shall combine the estimate in (18) —iii) with an estimate that we have previously obtained. Let I be a closed interval of the real axis and let $C(I)$ be the space of continuous real-valued functions defined on I. In [4] it is proved that if L is a linear positive operator defined on $C(I)$ and with values in the same space, when it is finite, and such that L preserves linear functions then the following inequality:

(19) $$|L(f, x) - f(x)| \leqslant \max\{7/4, 3/2 +$$
$$+ L((e_1 - xe_0)^2, x)/h^2\}\, \omega_2(f, h)$$

holds for any $f \in C(I)$, $h > 0$ and $x \in I$.

However, we mention that the proof of (19) uses neither the condition that $C(I)$ is included in the domain of L, but only the condition that the functions e_j $j = 0, 1, 2$ belong to this domain, nor the condition that the values of the operator L are continuous functions. Also the condition $f \in C(I)$ can be replaced by the weak condition $f \in \mathscr{F}_b(I)$.

If we fix $x = x_0$ and if we denote by F the functional $f \to L(f, x_0)$ from (18)—iii) and (19) we infer:

THEOREM 2. *Let F be a functional as in Definition 2 and we suppose that I is a closed interval. Then:*

(20) $|F(f) - f(x_0)| \leqslant \min\{3/2 + \rho^{-2}F(\psi), 1 + 3\rho^{-2}F(\Psi)\}\,\omega_2(f, \rho)$

for any $f \in C(I) \cap V$ and any real number $\rho > 0$.

2. Applications to the operators of S. N. Bernstein. The results in §1 give pointwise estimates for the linear positive operators that preserve linear functions. We illustrate the application of these estimates to the classical operators of S. N. Bernstein.

For any $n \in \mathbb{N}$ we denote by $B_n : \mathscr{F}_b[0,1] \to \mathscr{P}_n$ the operator defined by $B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, p_{nk}(x)$, $x \in [0,1]$ where $p_{nk}(x) = \binom{n}{k} x^k (1 - x)^{n-k}$ and $f \in \mathscr{F}_b[0,1]$.

If we fix $x = x_0$ the functional $f \to B_n(f, x_0)$ is a particular case of Definition 2 and then we can apply Corollary 1.

THEOREM 3. *For any $n \in \mathbb{N}$, $n \geqslant 1$ and any $f \in \mathscr{F}_b[0,1]$ we have*

(21) $$\|B_n(f) - f\| \leqslant 1{,}43\, \omega_2(f, n^{-\frac{1}{2}})$$

where $\|\cdot\|$ is the sup-norm.

Proof. From Corollary we obtain:

$$|B_n(f, x) - f(x)| \leqslant$$
$$\leqslant \left\{1 + \sum_{k=0}^{n} (2]\sqrt{n}|x - k/n|[+ (]\sqrt{n}|x - k/n|[)^2) p_{nk}(x)\right\} \omega_2(f, n^{-\frac{1}{2}}) \leqslant$$
$$\leqslant \{1 + {\sum_k}' (2\sqrt{n}|x - k/n| + n(x - k/n)^2)\, p_{nk}(x)\}\, \omega_2(f, n^{-\frac{1}{2}})$$

where ${\sum_k}'$ denotes the sum taken over those indices k for which $|x - k/n| > n^{-\frac{1}{2}}$. In [6] and [7] the following inequality is proved:

(22) $$\sqrt{n}\, {\sum_k}' |x - k/n| \cdot p_{nk}(x) \leqslant \eta - 1, \text{ where}$$

$$\eta = \frac{4306 + 837\sqrt{6}}{5832} \leqslant 1{,}09 \text{ is the Sikkema's constant.}$$

Hence $|B_n(f, x) - f(x)| \leqslant \{1 + 2(\eta - 1) + x(1 - x)\}\, \omega_2(f, n^{-\frac{1}{2}}) \leqslant$ $\leqslant 1{,}43\, \omega_2(f, n^{-\frac{1}{2}})$ for any $x \in [0,1]$.

Remark 6. The constant in Theorem 3 improves the constant equal to $1 \frac{72}{115} = 1{,}626\ldots$ given in [5].

THEOREM 4. *For any* $f \in \mathscr{F}_b[0,1]$ *that is not linear we have*:

$$\limsup_{n\to\infty} \frac{\| B_n(f) - f \|}{\omega_2(f, n^{-\frac{1}{2}})} \leqslant 1{,}30 \tag{23}$$

We shall use the following lemma. In what follows we use the notation $\binom{n}{m}$ in an extended sense for $n, m \in \mathbb{Z}$ that is $\binom{n}{m} = 0$ if $n < 0$ or $m < 0$ or $n < m$.

LEMMA 5. *For* $n \geqslant 2$, $0 \leqslant r \leqslant n$ *and* $0 \leqslant s \leqslant n$ *we have*:

$$\sum_{k=0}^{r} (x - k/n)^2 p_{nk}(x) = \binom{n-1}{r} \cdot x^{r+1}(1-x)^{n-r}\left(x - \frac{r}{n-1}\right) + \frac{x(1-x)}{n} \sum_{j=0}^{r-1} p_{n-2,j}(x), \tag{24}$$

$$\sum_{k=s}^{n} (x - k/n)^2 p_{nk}(x) = \binom{n-1}{s-1} \cdot x^{s}(1-x)^{n-s-1}\left(\frac{s-1}{n-1} - x\right) + \frac{x(1-x)}{n} \sum_{j=s-1}^{n-2} p_{n-2,j}(x), \tag{25}$$

where $x \in [0,1]$.

Proof. Let x and n fixed. We write $c_k = (x - k/n)^2 p_{nk}(x)$ and $d_k = \binom{n-2}{k-2} \cdot x^k (1-x)^{n-k+2}$ for $0 \leqslant k \leqslant n$.

We have:

$$c_k = \binom{n}{k} x^{k+2} (1-x)^{n-k} - 2\binom{n-1}{k-1} x^{k+1}(1-x)^{n-k} +$$
$$+ \frac{k}{n}\binom{n-1}{k-1} x^k (1-x)^{n-k} = \binom{n-2}{k} x^{k+2} (1-x)^{n-k} +$$
$$+ 2\binom{n-2}{k-1} x^{k+2} (1-x)^{n-k} + \binom{n-2}{k-2} x^{k+2} (1-x)^{n-k} -$$
$$- 2\binom{n-2}{k-1} x^{k+1} (1-x)^{n-k} - 2\binom{n-2}{k-2} \cdot x^{k+1} (1-x)^{n-k} +$$
$$+ \binom{n-2}{k-2} \cdot x^k (1-x)^{n-k} + \frac{1}{n}\binom{n-2}{k-1} \cdot x^k (1-x)^{n-k} =$$
$$= d_{k+2} - 2\, d_{k+1} + d_k + \frac{1}{n}\binom{n-2}{k-1} \cdot x^k (1-x)^{n-k}, \text{ because } d_0 = 0 = d_1.$$

Hence $\sum_{k=0}^{r} c_k = d_{r+2} - d_{r+1} + \frac{1}{n} \sum_{k=0}^{r} \binom{n-2}{k-1} x^k (1-x)^{n-k} =$

$$\binom{n-1}{r} x^{r+1}(1-x)^{n-r}\left(x - \frac{r}{n-1}\right) + \frac{1}{n}\sum_{k=0}^{r}\binom{n-2}{k-1} x^k (1-x)^{n-k}.$$

Therefore (24) is proved. If we take in (24) $x = 1 - y$, $r = n - s$, $k = n - p$ and $j = n - i - 2$ we obtain:

$$\sum_{p=s}^{n} (-y + p/n)^2 \binom{n}{n-p} \cdot (1-y)^{n-p} y^p =$$
$$= \binom{n-1}{n-s}(1-y)^{n-s+1} y^s \left(\frac{s-1}{n-1} - y\right) +$$
$$+ \frac{(1-y)y}{n} \sum_{i=s-1}^{n-2} \binom{n-2}{n-i-2} (1-y)^{n-i-2} y^i .$$

In the last equality putting $y = x$, $p = k$ and $i = j$ we obtain (25).

Proof of the theorem. From (18)—ii we obtain for $x \in [0,1]$ and $n \geqslant 3$:

$$|B_n(f, x) - f(x)| \leqslant$$
$$\leqslant \{1 + \textstyle\sum_k{}' (2\sqrt{n}\; |x - k/n| + n(x - k/n)^2)\, p_{nk}(x)\}\, \omega_2(f, n^{-\frac{1}{2}}).$$

where $\sum_k{}'$ denotes the sum taken over those indices k for which $|x - k/n| > n^{-\frac{1}{2}}$.

Let $r =]\, nx - \sqrt{n}[$ and $s = [nx + \sqrt{n} + 1]$. Hence $|x - k/n| > n^{-\frac{1}{2}}$ is equivalent with $k \leqslant r$ or $k \geqslant s$. In [6] the following equality is obtained:

$$\sum_k{}' \;|x - k/n|\; p_{nk}(x) = \tag{26}$$
$$= \binom{n-1}{r} x^{r+1}(1-x)^{n-r} + \binom{n-1}{s-1} x^s (1-x)^{n-s+1}.$$

From Lemma 5 we obtain:

$$\sum_k{}' n(x - k/n)^2 p_{nk}(x) = n\binom{n-1}{r} x^{r+1}(1-x)^{n-r}\left(x - \frac{r}{n-1}\right) +$$
$$+ n\binom{n-1}{s-1} x^s (1-x)^{n-s+1}\left(\frac{s-1}{n-1} - x\right) + \tag{27}$$
$$+ x(1-x)\left\{\sum_{j=0}^{r-1} p_{n-2,j}(x) + \sum_{j=s-1}^{n-2} p_{n-2,j}(x)\right\}$$

The equality $r =]nx - \sqrt{n}[$ means $n^{-\frac{1}{2}} + r/n < x \leqslant n^{-\frac{1}{2}} + (r+1)/n$.
We have $r/(n-1) \leqslant r/n + n^{-\frac{1}{2}}$ and $r/(n-1) + n^{-\frac{1}{2}} + n^{-1} \geqslant n^{-\frac{1}{2}} + (r+1)/n$ Then the following statement is true :

(28)
$$0 \leqslant x - r/(n-1) < n^{-\frac{1}{2}} + n^{-1}.$$

In a similar mode the equality $s = [nx + \sqrt{n} + 1]$ means :

$(s-1)/n - n^{-\frac{1}{2}} \leqslant x < s/n - n^{-\frac{1}{2}}$. We have $(s-1)/(n-1) \geqslant s/n - n^{-\frac{1}{2}}$

and $(s-1)/(n-1) - n^{-\frac{1}{2}} - n^{-1} \leqslant s - 1/n - n^{-\frac{1}{2}}$. Hence:

(29)
$$0 \leqslant (s-1)/(n-1) - x \leqslant n^{-\frac{1}{2}} + n^{-1}.$$

From (26), (28), (29) and (22) we obtain :

(30)
$$n\left\{\binom{n-1}{r} x^{r+1}(1-x)^{n-r}\left(x - \frac{r}{n-1}\right) + \binom{n-1}{s-1} \cdot x^{s}(1-x)^{n-s+1}\left(\frac{s-1}{n-1} - x\right)\right\} \leqslant$$
$$\leqslant \left(1 + n^{-\frac{1}{2}}(\eta - 1) = \eta - 1 + 0(1)(n \to \infty).\right.$$

From the inequality $(r-3)/(n-2) + (n-2)^{-\frac{1}{2}} \leqslant r/n + n^{-\frac{1}{2}} < x$ it results that for $0 \leqslant j \leqslant r-3$, $|x - j/(n-2)| > (n-2)^{-\frac{1}{2}}$, and from the inequality $s/(n-2) - (n-2)^{-\frac{1}{2}} \geqslant s/n - n^{-\frac{1}{2}} < x$ it results $|x - j/(n-2)| > (n-2^{-\frac{1}{2}})$ for $s \leqslant j \leqslant n-2$.

Then : $x(1-x)\left\{\sum_{j=0}^{r-1} p_{n-2\,j}(x) + \sum_{j=s-1}^{n-2} p_{n-2\,j}(x)\right\} \leqslant$

$$\leqslant x(1-x)\{\textstyle\sum_j'' p_{n-2\,j}(x) + p_{n-2\,r-2}(x) + p_{n-2\,r-1}(x) + p_{n-2\,s-1}(x)\} \leqslant$$

$$\leqslant x(1-x)\{\sqrt{n-2}\textstyle\sum_j'' |x - j/(n-2)| \cdot p_{n-2\,j}(x) +$$

$$+ 3 \cdot \max_{x\in[0,1]} \max_{0\leqslant j\leqslant n-2} p_{n-2\,j}(x)\}$$

where $\sum_j''$ means the sum taken over those indices j for which $|x - j/(n-2)| > (n-2)^{-\frac{1}{2}}$. By using Stirling's formula it can be proved that :

$$\max_{x\in[0,1]} \max_{0\leqslant j\leqslant n-2} p_{n-2\,j}(x) = 0(1), \ (n \to \infty).$$

Then by applying (22) for $n-2$ instead of n we obtain :

(31)
$$x(1-x)\left\{\sum_{j=0}^{r-1} p_{n-2\,j}(x) + \sum_{j=s-1}^{n-2} p_{n-2\,j}(x)\right\} \leqslant \frac{1}{4}(\eta - 1) + 0(1) \ (n \to \infty).$$

From (27), (30), (31) it follows :

(32)
$$\sum_k{}' n(x - k/n)^2 p_{n\,k}(x) \leqslant \frac{5}{4}(\eta - 1) + 0(1), \ (n \to \infty).$$

Finally, if we take into account (22) we obtain :

$$\| B_n(f) - f \| \leqslant \left(1 + \frac{13}{4}(\eta - 1) + 0(1)\right) \omega_2(f, n^{-\frac{1}{2}}), \quad (n \to \infty),$$

but we have $1 + \frac{13}{4}(\eta - 1) = 1{,}2928\ldots$

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