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CONVEX FUNCTIONS OF ORDER n AND P_n-SIMPLE FUNCTIONALS

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Abstract. A certain characterization of convex functions of order n on an interval I which are $(n+1)$ times differentiable on I, by the use of a P_n-simple functional L, is proved to be in connection with a certain behaviour of the functional L with respect to the strictly quasiconvex functions of order $n-1$. In context it is also proved that the necessary and sufficient condition for an n times continuously differentiable real function f be strictly quasiconvex of order $n(n\geqslant 0)$ on I is that f be convex or concave of order n on I, or there exist $c\in I$ such that f be concave of order n on $I\cap(-\infty, c]$ and convex of order n on $I\cap[c,+\infty)$.

1. Introduction. According to a result of H. T. Wang [8] if a function $f:(0,1)\to\mathbb{R}$ is convex of the first order (i.e., strictly convex) on $(0,1)$ then the Fourier coefficient

$$(1.1)\qquad a_1(f;\, s,t) = 2(t-s)^{-1}\int_s^t f(y)\cos\frac{2\pi(y-s)}{t-s}\,dy$$

is positive for any subinterval $[s, t]$, $s<t$, of $(0,1)$. Conversely, if the function f is twice differentiable on $(0,1)$ and $a_1(f;\, s,t)>0$ for all $s,t\in(0,1)$, $s<t$, then f is convex of the first order on $(0,1)$.

In the paper [7] we have observed that this characterization of the twice differentiable strictly convex functions may be done in terms of a P_1-simple functional L, namely $L:C[0,2\pi]\to\mathbb{R}$,

$$(1.2)\qquad L(f) = \int_0^{2\pi} f(x)\cos x\,dx,\quad f\in C[0,2\pi].$$

Indeed, for this functional one has

$$(1.3)\qquad L(D_{s,t}(f)) = \pi\, a_1(f;\, s,t)$$

for any $[s,t]\subset(0,1)$, where $D_{s,t}$ stands for the operator from $C[s,t]$ into $C[a,b]$,

$$(1.4)\qquad D_{s,t}(f)(x) = f(x(t-s)/(b-a)+(bs-at)/(b-a)),$$
$$x\in[a,b],\ f\in C[s,t].$$

Thus, the positivity of the Fourier coefficient (1.1) means

$$L(D_{s,t}(f)) > 0. \tag{1.5}$$

More generally, we have proved in [7] the following theorem.

THEOREM 1 ([7]). *Let $L : C[a, b] \to \mathbb{R}$ be a P_1-simple functional. The following statements are equivalent :*

1°. The necessary and sufficient condition that a twice differentiable real function f defined on an interval I be convex of the first order on I is that inequality (1.5) hold for all $s,t \in I$, $s < t$.

2°. For every real function f which is twice differentiable, strictly quasiconvex and nonmonotone on some interval $[s, t]$, there exist s' and t', $s \leqslant s' < t' \leqslant t$, such that

$$L(D_{s',t'}(f)) \geqslant 0. \tag{1.6}$$

As an application, other than the above result of H. T. Wang, we have mentioned the following corollary.

COROLLARY 1 ([7]). *In order that a twice differentiable real function f defined on an interval I be convex of the first order on I it is necessary and sufficient that*

$$\int_{-1}^{1} w(x)P_2^{(\alpha,\beta)}(x)f(t(1+x)/2 + s(1-x)/2)\mathrm{d}x > 0, \tag{1.7}$$

for all, s, $t \in I$, $s < t$.

Here we have denoted by $w(x)$ the function $(1-x)^{\alpha}$ $(1+x)^{\beta}$, $x \in (-1,1)$, where $\alpha > -1$, $\beta > -1$ and by $P_2^{(\alpha,\beta)}$ the Jacobi polynomial of second degree. In this case, the P_1-simple functional which intervenes is

$$L(f) = \int_{-1}^{1} w(x)P_2^{(\alpha,\beta)}(x)f(x)\, \mathrm{d}x, \qquad f \in C[-1,1]. \tag{1.8}$$

This paper is concerned with the extension of Theorem 1 to the case of convex functions of order n and P_n-simple functionals ($n > 1$).

2. **Preliminaries.** We first recall that a real function f defined in I is said to be *convex, nonconcave, polynomial, nonconvex, concave of order* n on I if the inequality

$$[x_1,x_2,\ldots,x_{n+2}\,; f] >,\ \geqslant,\ =,\ \leqslant,\ < 0 \tag{2.1}$$

is satisfied for every system of $n+2$ distinct points $x_1, x_2, \ldots, x_{n+2} \in I$. All these functions are said to be *of order* n on I.

Throughout we shall assume that the points $x_1,x_2,\ldots,x_{n+2}$ in a divided difference $[x_1, x_2,\ldots,x_{n+2}\,; f]$ are distinct and $x_1 < x_2 < \ldots < x_{n+2}$.

For $n = 0$ we have the monotone functions : increasing, nondecreasing, constant, nonincreasing and decreasing, respectively.

For $n = 1$ we have the convex, nonconcave, linear, nonconvex and concave functions, respectively.

Among the properties of the divided differences the following mean value theorem due to T. Popoviciu (see [2]) will be used frequently :

Let f be a real function defined on the points $x_1 < x_2 < \ldots < x_m$, where $m \geqslant n+2$. Then

$$[x_{i_1}, x_{i_2},\ldots, x_{i_{n+1}}\,; f] = \sum_{j=i_1}^{i_{n+1}-n} A_j[x_j, x_{j+1},\ldots, x_{j+n}\,; f], \tag{2.2}$$

where A_j are independent of f, $A_j \geqslant 0$, $j = i_1, i_1+1,\ldots, i_{n+1}-n$ and $\sum_{j=1_1}^{i_{n+1}-n} A_j = 1$.

Let us consider the interval $[a, b]$ and the integer $n \geqslant -1$. Denote by P_n the space of all polynomials of degree not greater than n ($P_{-1} = \{0\}$). Let $e_i(x) = x^i$, $x \in [a, b]$, $i = 0,1,\ldots$. Let S be a linear subspace of $C[a, b]$ which contains all polynomials. A linear functional $L : S \to \mathbb{R}$ is said to be *P_n-simple* if for every $f \in S$ there exist $n+2$ distinct points $t_1, t_2,\ldots,t_{n+2}$ in $[a, b]$ such that

$$L(f) = K[t_1,\ldots,t_{n+2}\,; f], \tag{2.3}$$

where K is a positive constant independent of f.

It is well known the following criterion of P_n-simplicity, due to T. Popoviciu : If the linear functional $L : S \to \mathbb{R}$ satisfies $L(e_i) = 0$, $i = 0,1,\ldots,n$ and $L(f) > 0$ for any function $f \in S$, convex of order n, then it is P_n-simple (see [2], Theorem 5.4.1).

Our paper deals with a kind of converse of this result.

For each interval $[s, t]$ consider the operator

$$D_{s,t} : C[s, t] \to C[a, b],$$

$$D_{s,t}(f)(x) = f(x(t-s)/(b-a) + (bs-at)/(b-a)),\quad x \in [a, b]. \tag{2.4}$$

Let $L : S \to \mathbb{R}$ be a P_n-simple functional and let $f : I \to \mathbb{R}$ be such that $D_{s,t}(f) \in S$ for every subinterval $[s,t]$ of I, $s < t$. It is clear that if f is convex of order n on I, then $L(D_{s,t}(f)) > 0$ for all $s, t \in I$, $s < t$.

It is natural to ask what property of a P_n-simple functional guarantees that the converse statement in the above proposition is also valid? We will show that in case that the function f is assumed to be in addition $(n+1)$ times differentiable, this property consists of a certain behaviour of the functional L with respect to the strictly quasiconvex functions of order $n-1$.

The quasiconvex functions of order n have been defined recently by E. Popoviciu [3]. A real function f defined in I is said to be *quasiconvex of order* n ($n \geqslant 0$) on I provided that the following condition

$$[x_2, x_3,\ldots,x_{n+2}\,; f] \leqslant \max\left([x_1, x_2,\ldots,x_{n+1}\,; f],\ [x_3, x_4,\ldots,x_{n+3}\,; f]\right) \tag{2.5}$$

holds for every system of $n+3$ points in $I : x_1 < x_2 < \ldots < x_{n+3}$. The function f is said to be *strictly quasiconvex of order* n on I if the inequality in (2.5) is always strict.

We remark that inequality (2.5) is equivalent with the inequality

$$(2.6) \qquad 0 \leqslant \max(-[x_1, x_2, \ldots, x_{n+2}; f], [x_2, x_3, \ldots, x_{n+3}; f]).$$

For $n = 0$ we shall use the terms: quasiconvex and strictly quasiconvex without mentioning the order 0.

3. Characterisation of smooth quasiconvex functions of order n.

LEMMA 1. *Let $f \in C^n(I)$ be (strictly) quasiconvex of order $n(n \geqslant 0)$ on the interval I. Then f is of order n (concave or convex of order n) on I, or there exists $c \in I$ such that f be nonconvex (concave) of order n on $I \cap (-\infty, c]$ and nonconcave (convex) of order n on $I \cap [c, +\infty)$.*

Proof. Suppose that $f \in C^n(I)$ is quasiconvex of order n on I. We will show that $f^{(n)}$ is of order zero on I or that there exists $c \in I$ such that $f^{(n)}$ be nonincreasing on $I \cap (-\infty, c]$ and nondecreasing on $I \cap [c, +\infty)$, whence the conclusion follows immediately. Assume, a contrario, that there are the points $a < b < c$ in I such that $f^{(n)}(a) < f^{(n)}(b) > f^{(n)}(c)$. Since $\lim [x_1, \ldots, x_{n+1}; f] = f^{(n)}(x)/n!$ for $x_i \to x$, $i = 1, \ldots, n+1$ (see [2], (5.2.17)*) we can find in I the points

$$a_1 < \ldots < a_{n+1} < b_1 < \ldots < b_{n+1} < c_1 < \ldots < c_{n+1}$$

such that $[a_1, \ldots, a_{n+1}; f] < [b_1, \ldots, b_{n+1}; f] > [c_1, \ldots, c_{n+1}; f]$. Whence it follows that there are $n+2$ consecutive points, say $x_1 < x_2 < \ldots < x_{n+2}$, between the points $a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}$ and also $n+2$ consecutive points, say $y_1 < y_2 < \ldots < y_{n+2}$, between the points $b_1, \ldots, b_{n+1}, c_1, \ldots, c_{n+1}$, such that $[x_1, x_2, \ldots, x_{n+2}; f] > 0$ and $[y_1, y_2, \ldots, y_{n+2}; f] < 0$. It is clear that from the points x_i at least x_1 is smaller than y_j, $j = 1, 2, \ldots, n+2$. The points x_i, $i = 1, 2, \ldots, n+2$ together with y_i, $i = 1, 2, \ldots, n+2$ determine the increasing sequence of points in I:

$$z_1 < z_2 < \ldots < z_k,$$

where $k \geqslant n+3$, $z_i = x_i$ and $z_{k-n-2+i} = y_i$ for $i = 1, 2, \ldots, n+2$. Since $[z_1, z_2, \ldots, z_{n+2}; f] > 0$ and $[z_{k-n-1}, z_{k-n}, \ldots, z_k; f] < 0$, there are $n+3$ points $z_{i_1}, z_{i_2}, \ldots, z_{i_{n+3}}$, $1 \leqslant i_1 < i_2 < \ldots < i_{n+3} \leqslant k$ such that $[z_{i_1}, z_{i_2}, \ldots, z_{i_{n+2}}; f] > 0$ and $[z_{i_2}, z_{i_3}, \ldots, z_{i_{n+3}}; f] < 0$, which contradicts the quasiconvexity of order n of f on I.

Next, if f is assumed to be strictly quasiconvex of order n on I, the remaining part of the proof is imediate.

LEMMA 2. *Let $f \in C^{n-1}(I)$ $(n \geqslant 1)$ be such that $f^{(n-1)}(c) = 0$ and $f^{(n-1)}(x) \neq 0$ for all $x \in I \cap (-\infty, c)$, where $c \in I$ is fixed. If $x_1 \in I$, $x_1 < c$, $x_1 < x_2^k < \ldots < x_{n+1}^k$ for $k = 1, 2, \ldots$ and $x_i^k \to c$ as $k \to \infty$, $i = 2, \ldots, n+1$, then zero is not a limit point of the sequence $([x_1, x_2^k, \ldots, x_{n+1}^k; f])_{k \geqslant 1}$.*

Proof. Passing if necessary from f to a certain $f + p_{n-2}$ where $p_{n-2} \in P_{n-2}$, we may assume that $f^{(r)}(c) = 0$ and $f^{(r)}(x) \neq 0$ for $x \in I \cap (-\infty, c)$, $r = 0, \ldots, n-1$.

Next we prove Lemma 2 by mathematical induction after n. For $n = 1$ the conclusion is trivial. If we assume it true for $n-1$, then

according to the recurrence formula

$$[x_1, x_1^k, \ldots, x_{n+1}^k; f] = (x_{n+1}^k - x_1)([x_2^k, \ldots, x_{n+1}^k; f] - [x_1, x_2^k, \ldots, x_n^k; f])$$

and since

$$[x_2^k, \ldots, x_{n+1}^k; f] \to f^{(n-1)}(c)/(n-1)! = 0 \text{ as } k \to \infty,$$

we immediately see that the conclusion is also true for n.

Remark 1. Similarly we can prove that if $f \in C^{n-1}(I)$ $(n \geqslant 1)$, $f^{(n-1)}(c) = 0$, $f^{(n-1)}(x) \neq 0$ for all $x \in I \cap (c, +\infty)$, $x_{n+1} \in I$, $c < x_{n+1}$, $x_1^k < \ldots < x_n^k < x_{n+1}$ for $k = 1, 2, \ldots$ and $x_i^k \to c$ as $k \to \infty$, $i = 1, \ldots, n$, then zero is not a limit point of the sequence $([x_1^k, \ldots, x_n^k, x_{n+1}; f])_{k \geqslant 1}$.

LEMMA 3. *Let $f \in C^n(I)$ $(n \geqslant 2)$ and $c \in I$. If f is nonconvex (concave) of order n on $I \cap (-\infty, c]$ and nonconcave (convex) of order n on $I \cap [c, +\infty)$ then f is (strictly) quasiconvex of order n on I.*

Proof. Assume that f is concave of order n on $I \cap (-\infty, c]$ and convex of order n on $I \cap [c, +\infty)$. Then $f^{(n)}$ is decreasing on the first interval and increasing on the second interval and there is a polynomial $p_n \in P_n$ such that

$$(f + p_n)^{(n)}(c) = 0, \ (f + p_n)^{(n-1)}(c) = 0 \text{ and } (f + p_n)^{(n-1)}(x) \neq 0, \text{ for every } x \neq c.$$

Thus $f + p_n$ satisfies the assumption of Lemma 2 and Remark 1. Therefore we may suppose that f is so that $f^{(n)}(c) = 0$, $f^{(n-1)}(c) = 0$ and $f^{(n-1)}(x) \neq 0$ for all $x \neq c$.

Now suppose that f is not strictly quasiconvex of order n on I. Then there are the points

$$(3.1) \qquad a_1 < a_2 < \ldots < a_{n+2} < a_{n+3}$$

in I, such that

$$(3.2) \qquad [a_1, a_2, \ldots, a_{n+2}; f] \geqslant 0 \text{ and } [a_2, a_3, \ldots, a_{n+3}; f] \leqslant 0.$$

Since f is concave of order n an $I \cap (-\infty, c]$ and convex of order n on $I \cap [c, +\infty)$; by (3.2), we see that $a_2 < c < a_{n+2}$. In case $a_k < c < a_{k+1}$ $(2 \leqslant k \leqslant n+1)$, by (3.2) and by the mean value theorem (2.2) of divided differences, we must have

(a)
$$[a_2, \ldots, a_k, c, a_{k+1}, \ldots, a_{n+2}; f] \geqslant 0 \text{ and}$$
$$[a_3, \ldots, a_k, c, a_{k+1}, \ldots, a_{n+3}; f] \leqslant 0,$$

or

(b)
$$[a_2, \ldots, a_k, c, a_{k+1}, \ldots, a_{n+2}; f] \leqslant 0 \text{ and}$$
$$[a_1, \ldots, a_k, c, a_{k+1}, \ldots, a_{n+1}; f] \geqslant 0.$$

Therefore, replacing if necessary points (3.1) with

$$a_2, \ldots, a_k, c, a_{k+1}, \ldots, a_{n+3} \text{ in case (a)}$$

or with

$$a_1, \ldots, a_k, c, a_{k+1}, \ldots, a_{n+2} \text{ in case (b)},$$

we may assume that in (3.1), (3.2) we have $a_k = c$ for a certain k, $3 \leqslant k \leqslant n + 1$.

In what follows we show that all points in (3.1), except c, can be replaced by arbitrary points of the interval (a_{k-1}, a_{k+1}) after a predicted number of steps.

First step: Let $x_1 \in (a_{k-1}, c)$ be arbitrarily chosen. We may include x_1 in system (3.1) omitting a_1 or a_{n+3}, such that the new system of points (3.1) also satisfies relation (3.2). This can be proved in the same way by that it was shown that c can be included in (3.1) with preservation of (3.2).

Second step: Include in (3.1) a point $y_1 \in (c, a_{k+1})$ instead of a_1 or a_{n+3}, as at first step.

Then include in (3.1) $x_2 \in (x_1, c)$, $y_2 \in (c, y_1)$ and so on (such that condition (3.2) be fulfilled after each step).

It is not difficult to see that after at most $2N$ steps (N depending only on n) from the initial points a_i there will remain in (3.1) only c and a_{k-1} or c and a_{k+1}.

Now let us consider the sequence $(x_i^p)_{p\geqslant 1}$ and $(y_i^p)_{p\geqslant 1}$, $i = 1, 2, \ldots, N$, such that

$$a_{k-1} < x_1^p < x_2^p < \ldots < x_N^p < c < y_N^p < \ldots < y_2^p < y_1^p < a_{k+1} \tag{3.3}$$

for all $p = 1, 2, \ldots,$ and

$$x_1^p \to c,\ y_1^p \to c \text{ as } p \to \infty. \tag{3.4}$$

For each p apply to points (3.1) the above algorithm with $x_1 = x_1^p$, $x_2 = x_2^p, \ldots;\ y_1 = y_1^p,\ y_2 = y_2^p, \ldots$. We obtain the points

$$a_1^p < a_2^p < \ldots < a_{n+2}^p < a_{n+3}^p \tag{3.5}$$

satisfying (3.2), such that $a_1^p = a_{k-1}$ (or $a_{n+3}^p = a_{k+1}$) and

$a_j^p \in \{x_i^p : i = 1, 2, \ldots, N\} \cup \{y_i^p : i = 1, 2, \ldots, N\}$ for any $j = 2, 3, \ldots, n + 3$ (respectively $j = 1, 2, \ldots, n + 2$). Passing if necessary to a subsequence we may assume that $a_1^p = a_{k-1}$ for all p or that $a_{n+3}^p = a_{k+1}$ for all p.

In the first case: $a_1^p = a_{k-1}$ for all p, the first inequality in (3.2) yields

$$[a_{k-1}, a_2^p, \ldots, a_{n+1}^p; f] \leqslant [a_2^p, a_3^p, \ldots, a_{n+2}^p; f],\ p = 1,2,\ldots. \tag{3.6}$$

Since $a_j^p \to c$ as $p \to \infty$, $j = 2, 3, \ldots, n + 2$, the right-hand side in (3.6) tends to $f^{(n)}(c)/n! = 0$. Then by Lemma 2 we conclude that each limit point of the left-hand side is < 0. But this is impossible because $f^{(n)}(x) > 0$ for every $x \neq c$. This contradiction shows that f must be strictly quasiconvex of order n on I.

If f is assumed nonconvex (nonconcave) of order n on $I \cap (-\infty, c]$ ($I \cap [c, +\infty)$) then, by a somewhat similar reasoning we can prove that f is quasiconvex of order n on I. We omit the details.

Remark 2. The conclusion of Lemma 3 is valid for $n = 0$ and $n = 1$ without any assumption on the smoothness of f.

This assertion is trivial if $n = 0$. Let $n = 1$ and f be nonconvex (concave) on $I \cap (-\infty, c]$ and nonconcave (convex) on $I \cap [c, +\infty)$.

Supose that f is not (strictly) quasiconvex of first order on I. Then there are points $a_1 < a_2 < a_3 < a_4$ in I such that

$$[a_1, a_2, a_3; f] >(\geqslant) 0 \text{ and } [a_2, a_3, a_4; f] < (\leqslant) 0. \tag{3.7}$$

Whence we see that $a_2 < c < a_3$. Again by (3.7) using inequalities

$$[a_1, a_2, c; f] \leqslant (<)\ 0 \text{ and } [c, a_3, a_4; f] \geqslant (>)\ 0$$

and the mean value theorem (2.2) of divided differences, we derive both $[a_2, c, a_3; f] > 0$ and $[a_2, c, a_3; f] > 0$, a contradiction. Thus f must be (strictly) quasiconvex of first order on I.

THEOREM 2. *In order that a function $f \in C^n(I)$ $(n \geqslant 0)$ be (strictly) quasiconvex of order n on the interval I it is necessary and sufficient that f be of order n (concave or convex of order n) on I or there exist a point $c \in I$ such that f be nonconvex (concave) of order n on $I \cap (-\infty, c]$ and nonconcave (convex) of order n on $I \cap [c, +\infty)$.*

Proof. See Lemma 1, Lemma 3 and Remark 2.

Counterexample 1. Let $n = 0$, $I = [0,1]$, $f : I \to \mathbb{R}$, $f(0) = 1$, $f(x) = x$ for any $x \in (0, 1]$.

This function is strictly quasiconvex on I but there is not $c \in I$ such that f be decreasing on $[0, c]$ and increasing on $[c, 1]$. This shows that the assumption that $f \in C(I)$ in Lemma 1 is essential.

Counterexample 2. Let $n = 2$, $I = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x(x+2)^2$ for $x \leqslant 0$, $f(x) = x(x-2)^2$ for $x > 0$.

This function is concave of order 2 on $(-\infty, 0]$ and convex of order 2 on $[0, +\infty)$. Nevertheless it is not quasiconvex of order 2 on $\mathbb{R}$ because $[-2, -1, 0, 1; f] > 0$ and $[-1, 0, 1, 2; f] < 0$. This shows that only continuity does not suffice for that Lemma 3 apply.

4. **The main result.** Let $[a, b]$ be a fixed interval and S be a linear subspace of $C[a, b]$ which contains all $(n + 1)$ times differentiable functions defined on $[a, b]$.

THEOREM 3. *Let $L : S \to \mathbb{R}$ be a P_n-simple functional $(n \geqslant 1)$. The following statements are equivalent*:

1°. *The necessary and sufficient condition that a $(n + 1)$ times differentiable real function f defined on some interval I be convex of order n on I is that*

$$L(D_{s,t}(f)) > 0, \tag{4.1}$$

for all $s, t \in I$, $s < t$.

2. *For any $(n + 1)$ times differentiable real function f defined on some interval $[s, t]$, strictly quasiconvex of order $n - 1$, which is not of order $n - 1$ on $[s, t]$, there exist s' and t', $s \leqslant s' < t' \leqslant t$, such that*

$$L(D_{s',t'}(f)) \geqslant 0. \tag{4.2}$$

Proof. 1° ⇒ 2°. Assume that 2° is not true, that is, there exists an $(n + 1)$ times differentiable real function f defined on some interval $[s, t]$, strictly quasiconvex of order $n - 1$, which is not of order $n - 1$

on $[s, t]$, such that

(4.3) $$L(D_{s',t'}(f)) < 0 \quad \text{for all } s', t' \in [s, t],\ s' < t'.$$

Then, by Theorem 2, there exists $c \in (s, t)$ such that f be concave of order $n - 1$ on $[s, c]$ and convex of order $n - 1$ on $[c, t]$. Consequently, each divided difference on $n + 1$ distinct points in $[s, c]$ being negative, is smaller than any divided difference on $n + 1$ distinct points in $[c, t]$ (the last one being positive). On the other hand, by (4.3) and 1°, $(-f)$ must be convex of order n on $[s, t]$. Hence

$$[x_1, x_2, \dots, x_{n+1}; f] > [x_2, x_3, \dots, x_{n+2}; f].$$

whenever $s \leqslant x_1 < x_2 < \dots < x_{n+2} \leqslant t$. Whence, it follows that each divided difference on $n + 1$ distinct points of $[s, c]$ is greater than any divided difference of the same kind on points of $[c, t]$, a contradiction. Thus, 1° ⇒ 2°.

Conversely, assume now that we have 2°. Let $f : I \to \mathbb{R}$ be a $(n + 1)$ times differentiable function such that condition (4.1) is satisfied for every $[s, t] \subset I$. In view of the strict inequality in (4.1) it will be sufficient to prove that f is nonconcave of order n on I. Assume by contradiction that there is not the case. Then there exists $c \in \operatorname{int} I$ such that $f^{(n+1)}(c) < 0$. Consequently, there is a number $r > 0$ such that $f^{(n)}(x) > f^{(n)}(c)$ for all $x \in [c - r, c) \subset I$ and $f^{(n)}(x) < f^{(n)}(c)$ for all $x \in (c, c + r] \subset I$. It follows that there is a polynomial $h \in P_n$ such that the function $g = f - h$ be convex of order $n - 1$ on $[c - r, c]$ and concave of the same order on $[c, c + r]$. Then, again by Theorem 2 $(-g)$ is strictly quasiconvex of order $n - 1$ on $[c - r, c + r]$ and is not of order $n - 1$ on this interval. Therefore, by 2°, there exist s' and t', $c - r \leqslant s' < t' \leqslant c + r$ such that $L(D_{s',t'}(-g)) \geqslant 0$, that is $L(D_{s',t'}(f)) \leqslant 0$, which contradicts (4.1). Hence f is nonconcave of order n on I as claimed and so 2° ⇒ 1°. The proof is now complete.

Remark 3. In order that an $(n + 1)$ times *continuously* differentiable real function f defined on an interval I be convex of order n on I, it is necessary and sufficient that inequality (4.1) holds for all $s, t \in I$, $s < t$.

Indeed, suppose that (4.1) holds for all $s, t \in I$, $s < t$. For each $[s, t]$ there is $c \in [a, b]$ such that, by (2.3) and (2.4), we have

$$L(D_{s,t}(f)) = (K/(n+1)!)D_{s,t}(f)^{(n+1)}(c) =$$

$$= (K/(n+1)!)\left(\frac{t-s}{b-a}\right)^{n+1} f^{(n+1)}(\theta),$$

where $\theta = c(t - s)/(b - a) + (bs - at)/(b - a) \in [s, t]$. Then $f^{(n+1)}(\theta) > 0$. This means that in each subinterval of I, $f^{(n+1)}$ takes positive values. Consequently, $f^{(n+1)}(x) \geqslant 0$ for all $x \in I$, that is f is nonconcave of order n on I, whence, again by (4.1), it follows that f is convex of order n on I.

5. **Examples.** a) Let $a = a_1 < a_2 < \dots < a_{n+2} = b$ be fixed. Consider the P_n-simple functional $(n \geqslant 1)$

(5.1) $$L : C[a, b] \to \mathbb{R},\ L(f) = [a_1, a_2, \dots, a_{n+2}; f].$$

We will show that for this functional proposition 2° in Theorem 3 is true. To this end, let us consider an $(n + 1)$ times differentiable real function f defined on the interval $[s, t]$, which is strictly quasiconvex of order $n - 1$ without being of order $n - 1$ on $[s, t]$. Then, by Theorem 2, there is a point $c \in (s, t)$ such that f is concave of order $n - 1$ on $[s, c]$ and convex of order $n - 1$ on $[c, t]$. Let $h = \min(c - s, t - c)$. Since $D_{s,s+h}(f)$ is concave of order $n - 1$ on $[a, b]$, we have

$$[a_1, a_2, a_4, \dots, a_{n+2};\ D_{s,s+h}(f)] < 0.$$

Similarly,

$$[a_1, a_2, a_4, \dots, a_{n+2};\ D_{c,c+h}(f)] > 0.$$

Consequently, there exists $s' \in (s, c)$ such that

$$[a_1, a_2, a_4, \dots, a_{n+2};\ D_{s',t'}(f)] = 0,$$

where $t' = s' + h$. Whence, by using the strict quasiconvexity of order $n - 1$ of f, we easily see that

$$[a_1, a_2, a_3, \dots, a_{n+2};\ D_{s',t'}(f)] \geqslant 0.$$

Thus inequality (4.2) holds

COROLLARY 2. *Let $a = a_1 < a_2 < \dots < a_{n+2} = b$ be fixed. In order that a $(n + 1)$ times differentiable real function f defined on an interval I be convex of order n on I it is necessary and sufficient that*

$$[a_1, a_2, \dots, a_{n+2};\ D_{s,t}(f)] > 0,$$

for all $s, t \in I$, $s < t$.

b) Let $\alpha > -1$, $\beta > -1$, $w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, $x \in (-1, 1)$ and let $P_{n+1}^{(\alpha,\beta)}$ denote the Jacobi polynomial of degree $n + 1$ $(n \geqslant 1)$. Consider the functional

(5.2) $$L : C[-1, 1] \to \mathbb{R},\ L(f) = \int_{-1}^{1} w(x)\, P_{n+1}^{(\alpha,\beta)}(x) f(x)\, dx.$$

As it has been shown by A. Lupaş [1], functional (5.2) is P_n-simple. Next we will prove that for this functional proposition 2° in Theorem 3 is true. Let f be an $(n + 1)$ times differentiable real function defined on $[s, t]$, strictly quasiconvex of order $n - 1$ without being of order $n - 1$ on $[s, t]$. There exists $c \in (s, t)$ such that f is concave of order $n - 1$ on $[s, c]$ and convex of order $n - 1$ on $[c, t]$. Denote by $j_1, j_2, \dots, j_{n+1}$ ($j_i < j_k$ for $i < k$) the roots of the Jacobi polynomial $P_{n+1}^{(\alpha,\beta)}$ and let $h = \min((c - s)/(j_{n+1} + 1), (t - c)/(1 - j_1))$.
For each $y \in [c - h(j_{n+1} - j_1), c]$ the points :

$$x_k(y) = y + h(j_k - j_1), \quad k = 1, 2, \dots, n + 1$$

belong to $[s, t]$ In addition, the divided difference

(5.3) $$[x_1(y), x_2(y), \dots, x_{n+1}(y); f]$$

is negative for $y = y_1 = c - h\,(j_{n+1} - j_1)$ and positive for $y = y_2 = c$. Using the continuity of (5.3) with respect to y, we see that there exists $y_0 \in (y_1, y_2)$ such that for $y = y_0$ the divided difference (5.3) be null. Let x_k, $k = 1,2,\ldots,n+1$ denote for briefness the points $x_k(y_0)$. Recalling that $h \leqslant (c - s)/(j_{n+1} + 1)$, the inequality $c - h(j_{n+1} - j_1) < x_1$ implies the existence of a point $s' \in (s, x_1)$ such that $(x_1 - s')/(j_1 + 1) = h$. Similarly, there exists $t' \in (x_{n+1}, t)$ such that $(t' - x_{n+1})/(1 - j_{n+1}) = h$. Now, since

$$\frac{x_1 - s'}{j_1 + 1} = \frac{x_2 - x_1}{j_2 - j_1} = \ldots = \frac{x_{n+1} - x_n}{j_{n+1} - j_n} = \frac{t' - x_{n+1}}{1 - j_{n+1}} = h,$$

we immediately see that $D_{s',t'}(f - p)(x)P_{n+1}^{(\alpha,\beta)}(x) \geqslant 0$ for all $x \in [-1, 1]$, where $p \in P_{n-1}$ and $p(x_k) = f(x_k)$, $k = 1, 2,\ldots, n+1$ (recall that $[x_1, x_2,\ldots,x_{n+1}; f] = 0$).

Consequently, $L(D_{s',t'}(f)) \geqslant 0$ as claimed.

COROLLARY 3. *In order that a $(n+1)$ times differentiable real function f defined on an interval I be convex of order n on I it is necessary and sufficient that*

$$\int_s^t w\left(\frac{2x - t - s}{t - s}\right) P_{n+1}^{(\alpha,\beta)}\left(\frac{2x - t - s}{t - s}\right) f(x)\,dx > 0$$

for all $s, t \in I$, $s < t$.

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