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ADAPTIVE SPLINE DIFFERENCE SCHEME FOR SINGULAR PERTURBATION PROBLEM WITH MIXED BOUNDARY CONDITIONS

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Abstract. The spline difference scheme derived in [3] using adaptive spline function approximation is applied to the self-adjoint perturbation problem with mixed boundary conditions. The error estimate uniformly in ε is analyzed. It is proved that scheme (4–6) preserves optimal error estimate of $O(h \min(h, \sqrt{\varepsilon}))$ over the uniform mesh, when $\beta_2 = 0$. Numerical experiments exhibiting expected uniform order are presented.

1. Introduction. In [3] is derived an exponential spline difference scheme using adaptive spline function approximation. That spline has an exponential base and in a case of homogeneous boundary conditions it gives a good approximation for the exact solution. Adaptive spline is derived to suit a self-adjoint singularly perturbed two-point boundary value problem. In [3] it is shown experimentally that adaptive spline difference scheme achieves a second order of uniform accuracy in a small parameter ε. In [4] is proved the second order of the uniform convergence, in a case of Dirichlet's boundary conditions even optimal convergence for that scheme when parameter q is variable, and when $p(x) = \text{const}$ a second order of uniform global convergence.

In this paper we consider another case of practical use: equation

$$-\varepsilon u'' + p(x)\, u = f(x) \tag{1}$$

along with the mixed boundary conditions:

$$\alpha_1 u(0) + \beta_1 u'(0) = \gamma_1 \qquad \alpha_1\beta_1 \leqslant 0, \ |\alpha_1| + |\beta_1| \neq 0 \tag{2}$$

$$\alpha_2 u(1) + \beta_2 u'(1) = \gamma_2 \qquad \alpha_2\beta_2 \geqslant 0, \ |\alpha_2| + |\beta_2| \neq 0 \tag{3}$$

where ε is a small parameter in $[0,1]$, $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2$, are given constants, $p(x)$, $q(x) \in C^2[0,1]$, $p(x) \geqslant p > 0$.

Problem with mixed boundary conditions concerned with fluid flow, elctrolysis, semiconductor device modeling and chemical reactions.

As usual for this kind of boundary conditions we have chosen approximations the order of which is one lower than for the inner equation.

In this paper it will be shown that our adaptive spline difference scheme (4 — 6) preserves the second order of uniform convergence in a small parameter ε when $\beta_2 = 0$ even optimal convergence in a sense of Miller [2]. But in a case of $\beta_2 \neq 0$ the order drops to one.

2. **The difference scheme.** Define the uniform mesh length $h = 1/N$, N is a positive integer. The mesh points are given by $x_i = ih$, $i = 0(1)n$. Denote by u_i the spline approximation to the solution at the point x_i. The scheme [3] will be written in a form:

$$r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = f_{i-1} q_i^- + f_i q_i^c + f_{i+1} q_i^+,$$

For $i = 1(1)N - 1$ with

$$r_i^- = 1 - \frac{h^2}{q^2}\frac{p_{i-1}}{\varepsilon}\left(1 - \frac{q}{\operatorname{sh} q}\right) \quad q_i^- = -\frac{1}{\varepsilon}\frac{h^2}{q^2}\left(1 - \frac{q}{\operatorname{sh} q}\right)$$

$$r_i^+ = 1 - \frac{h^2}{q^2}\frac{p_{i+1}}{\varepsilon}\left(1 - \frac{q}{\operatorname{sh} q}\right) \quad q_i^+ = -\frac{1}{\varepsilon}\frac{h^2}{q^2}\left(1 - \frac{q}{\operatorname{sh} q}\right)$$

$$r_i^c = -2 - 2\frac{h^2}{q^2}\frac{p_i}{\varepsilon}(-1 + q \operatorname{cth} q) \quad q_i^c = -\frac{2}{\varepsilon}\frac{h^2}{q^2}(-1 + q \operatorname{cth} q)$$

or in shortened notation

(4) $\quad R_i u_i = Q_i f_i \quad i = 1(1)N - 1 \quad q = \sqrt{\frac{p_i h}{\varepsilon}}, \ p = \text{const}, \ p(x_i) = p_i$

The following additional relations for $i = 0$ and $i = N$ are derived from the boundary conditions (2) — (3) (see [3]):

(5)
$$u_0\left\{\alpha_1 + \beta_1\left[\frac{\rho_0}{\varepsilon}\left(-\frac{h}{q}\operatorname{cth} q + \frac{h}{q^2}\right) - \frac{1}{h}\right]\right\} + u_1\left\{\beta_1\left[\frac{\rho_1}{\varepsilon}\left(\frac{h}{q \operatorname{sh} q} - \frac{h}{q^2}\right) + \frac{1}{h}\right]\right\} = \beta_1\left[\frac{f_1}{\varepsilon}\left(\frac{h}{q \operatorname{sh} q} - \frac{h}{q^2}\right) + \frac{f_0}{\varepsilon}\left(-\frac{h}{q}\operatorname{cth} q + \frac{h}{q^2}\right)\right] + \gamma_1$$

(6)
$$u_{n-1}\left\{\beta_2\left[\frac{p_{n-1}}{\varepsilon}\left(-\frac{h}{q \operatorname{sh} q} + \frac{h}{q^2}\right) - \frac{1}{h}\right]\right\} + u_n\left\{\alpha_2 + \beta_2\left[\frac{p_n}{\varepsilon}\left(\frac{h}{q}\operatorname{cth} q - \frac{h}{q^2}\right) + \frac{1}{h}\right]\right\} = \gamma_2 + \beta_2\left[\frac{f_n}{\varepsilon}\left(\frac{h}{q}\operatorname{cth} q - \frac{h}{q^2}\right) + \frac{f_{n-1}}{\varepsilon}\left(-\frac{h}{q \operatorname{sh} q} + \frac{h}{q^2}\right)\right]$$

The error analysis of (4) for variable parameter $q_i = \sqrt{\frac{p_{ih}}{\varepsilon}}$, $p_i = p(x_i)$ for the following boundary conditions $u(0) = \alpha_0$, $u(1) = \alpha_1$ is given in [4]. It is shown that when $i = 1\ (1)\ N - 1$ the following estimate holds

(7) $\quad |u(x_i) - u_i| \leq M\, h \min(h, \sqrt{\varepsilon}), \ \varepsilon \in (0,1], \ h < 1$

Here it will be given proof for the first and the last equation (5) and (6) for $i = 0$ and $i = N$. First, we consider the case $i = 0$. Then,

$$R^h u_0 = u_0\{\alpha_1 + \beta_1/h(-\rho_0 \operatorname{cth} \rho_0)\} + u_1\{\beta_1/h(\rho_1/sh\, \rho_1)\} = \gamma_1 +$$
$$+ (\beta_1/h)\{f_1/p_1(\rho_1/\operatorname{sh} \rho_1 - 1) + f_0/p_0(-\rho_0 \operatorname{cth} \rho_0)\}$$

for variable parameter $\rho_1 = \sqrt{\frac{p_1}{\varepsilon}}\, h$.

The matrix form of the scheme (4) — (6) is

$$Au = F$$

where $u = [u_0, u_1, \ldots, u_N]^T$ and $F = [f_0, f_1, \ldots, f_N]^T$, are vectors and A is a matrix of the system of equations.

The truncation error of the scheme (4) —(6) is defined as $\tau_i(u) = R(u(x_i) - u_i)$. We obtain the error at the nodes as

(8) $$|u(x_i) - u_i| \leq |A|^{-1} \max_i |\tau_i(u)|, \quad i = 0(1)n,$$

where u_i is an approximation for (1) at x_i.

So we must estimate norm of the matrix A and maximum of the truncation error of the scheme (4) — (6).

In the error analysis of this scheme we'll use the asymptotical expansion of the exact solution given according to [2].

Lemma 1. [2] *Let $u(x) \in C^4[0,1]$. Let $p'(0) = p'(1) = 0$. Then the solution of the problem (1) has the form*

(9) $$u(x) = u_0(x) + w_0(x) + g(x)$$

where

(10)
$$u_0(x) = q_0 \exp(-x(p(0)/\varepsilon)^{1/2}),$$
$$w_0(x) = q_1 \exp(-(1 - x)(p(1)/\varepsilon)^{1/2})$$

q_0 and q_1 are bounded functions of ε independent of x and

(11) $$|g^{(i)}(x)| \leq M(1 + \varepsilon^{1-i/2}), \ i = 0\ (1)\ n$$

M is a constant independent of ε.

Estimate of the matrix:

$$\Delta = r_0^+ + r_0^c = \left[\frac{\beta_1}{h}\left(-\rho_0 \operatorname{cth} \rho_0 + \frac{\rho_1}{\operatorname{sh} \rho_1}\right)\right]$$

By estimating this we obtain

(12)
(13)
$$|\Delta| \leq \begin{cases} Mh/\varepsilon & \text{for } h^2 \leq \varepsilon \\ M/\sqrt{\varepsilon} & \text{for } h^2 \geq \varepsilon \end{cases}$$

Estimate of the truncation error. According to Lemma 1 the truncation error of $u(x)$ is the sum of the truncation errors of the functions u_0, w_0, g, that is,

$$\tau_i(u) = \tau_i(u_0) + \tau_i(w_0) + \tau_i(g), \qquad i = 0(1)\,N.$$

Each of the summands of $\tau_i(u)$ will be estimated separately. We will start with u_0. The truncation error for the boundary layer function u_0 for $i = 0$ is

$$\tau_0(u_0) = \{r_0^c \exp(-\sqrt{p_0/\varepsilon}\, x_0) + r_0^+ \exp(-\sqrt{p_0/\varepsilon}\, x_1) -$$

$$-(p_0 - p_1)\exp(-\sqrt{p_0/\varepsilon}\, x_1)\, q_i^+ - \gamma_1)$$

From the boundary condition (2) we obtain

$$\alpha_1 = (\beta_1/h)\,\rho_0 + \gamma_1 \exp(-\sqrt{p(0)/\varepsilon}\, x_0)$$

where $\rho_0 = \sqrt{p(0)/\varepsilon}\, h$.

It simplifies the truncation error to:

$$\tau_0(u_0) = \exp(-\sqrt{p_0/\varepsilon}\, x_0)\ [(\beta_1/h)\,\rho_0 - (\beta_1/h)\,(\rho_0 \operatorname{cht} \rho_0)] +$$

$$+\exp(-\sqrt{p_0/\varepsilon}x_1)\,(\beta_1/h)\,(\rho_1/\operatorname{sh}\rho_1) - (p_0 - p_1)/p_1(\beta_1/h)$$

$$(\rho_1/\operatorname{sh}\rho_1 - 1)\exp(-\sqrt{p_0/\varepsilon}\, x_1), \text{ i.e.}$$

$$\tau_i(u_0) = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}x_0)\ \{\rho_0 - \rho_0 \operatorname{cth}\rho_0 + \exp(-\rho_0)\ \rho_1/\operatorname{sh}\rho_1 -$$

$$-(p_0 - p_1)/p_1 \exp(-\rho_0)\,(\rho_1/\operatorname{sh}\rho_1 - 1)\}$$

We will divide this expression in two parts τ_r and τ_q which we shall observe separately.

(14) $\quad \tau_r = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\, x_0)\,[\rho_0 - \rho_0 \operatorname{cth}\rho_0 + \exp(-\rho_0)\rho_1/\operatorname{sh}\rho_1]$

(15) $\quad \tau_q = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}x_0)\,[(p_0 - p_1)/p_1 \exp(-\rho_0)\ (\rho_1/\operatorname{sh}\rho_1 - 1)]$

$$\text{As } -\exp(-\rho_0) = \operatorname{sh}\rho_0 - \operatorname{ch}\rho_0$$

$$\tau_r = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\, x_0)\left[\rho_0 - \operatorname{sh}\rho_0 \frac{\rho_1}{\operatorname{sh}\rho_1} - \rho_0 \operatorname{cth}\rho_0 + \operatorname{ch}\rho_0 \frac{\rho_1}{\operatorname{sh}\rho_1}\right]$$

After Taylor's expansions of these expressions we obtain:

$$\tau_r = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\, x_0)\frac{\rho_0^2 - \rho_1^2}{6}\left\{1 - \rho_0 + \frac{\rho_0^2}{2} - \frac{7}{360}(\rho_0^2 + \rho_1^2) + 0(\rho_0^3)\right\}$$

$$\tau_q = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\, x_0)\ \frac{\rho_0^2 - \rho_1^2}{6}\left\{1 - \rho_0 + \frac{1}{2}\rho_0^2 - \frac{7}{60}\rho_1^2 + 0(\rho_0^3)\right\}$$

In the difference $\tau = \tau_r - \tau_q$ the hardest parts are cancelled. So we obtain:

$$\tau_0(u_0) = (\beta_1/h)\exp(-\sqrt{\rho_0/\varepsilon}\, x_0)\ \frac{\rho_0^2 - \rho_1^2}{6}\cdot 0(\rho_0^2), \text{ i.e.}$$

$$\tau_0(u_0) = (\beta_1/h)\exp(-\sqrt{\rho_0/\varepsilon}\, x_0)\ \frac{1}{6}\frac{h^2}{\varepsilon}(p(0) - p(x_1))\cdot 0(\rho_0^2)$$

Using estimate

(a) $\qquad |p(0) - p(x_1)| \leqslant M\, x_1^2$, since $p'(0) = 0$,

(b) $\qquad |t^k \exp(-t)| \leqslant C(\theta)\exp(-\theta t)$, $0 < \theta < 1$

we obtain the final estimate for $\tau_0(u_0)$

(16) $\quad |\tau_0(u_0)| \leqslant Mh^3/\varepsilon \exp(-\theta\sqrt{p(0)/\varepsilon}\, x_0)$, $0 < \theta < 1$, when $h^2 \leqslant \varepsilon$

From (7), (12) and (16) we obtain for boundary layer function $\tau_0(u_0)$ the following estimate

(17) $\qquad |u(x_0) - u_{(0)}| \leqslant Mh^2$ for $h^2 \leqslant \varepsilon$

When $h^2 \geqslant \varepsilon$ using estimate

$$\operatorname{sh}\rho_1/\rho_1 = \operatorname{sh}\rho_0/\rho_0 + M\,h\,x_1^2/\sqrt{\varepsilon}\exp(\delta\,\rho_0)\ \ 0 < \delta < 1$$

(14) yields

$$\tau_r = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\, x_0)\,[\rho_0 - \rho_0 \operatorname{cth}\rho_0 + \exp(-\rho_0)\rho_0/\operatorname{sh}\rho_0 +$$

$$+\exp(-\rho_0)\ Mh\,x_1^2/\sqrt{\varepsilon}\exp(\delta\, x_0)]$$

As $\exp(-\rho_0) = -(\operatorname{sh}\rho_0 - \operatorname{ch}\rho_0)$ we obtain

$$|\tau_r| \leqslant (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\, x_0)\ M\,h\,x_1^2/\sqrt{\varepsilon}\exp(-\rho_0)\exp(\delta\,\rho_0),$$

$$0 < \delta < 1, \text{ i.e. For } h^2 \geqslant \varepsilon$$

(18) $\qquad |\tau_r| \leqslant M\sqrt{\varepsilon}\exp(-\beta\, x_0/\sqrt{\varepsilon})$

where β is the smallest constant which appears in the proof.

Estimate for (15) is the following

$$\tau_q = (\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\,x_1)\ (p_0 - p_1)/p_1(\rho_1/\operatorname{sh}\rho_1 - 1)]$$

As $|\rho_1/\operatorname{sh}\rho_1| \leqslant M$

(19) $\quad |\tau_q| \leqslant M\, x_1^2(\beta_1/h)\exp(-\sqrt{p_0/\varepsilon}\, x_1) \leqslant M\,\varepsilon/h\exp(-\delta_1 x_1/\sqrt{\varepsilon})$,
$0 < \delta_1 < 1$ in a case $h^2 \geqslant \varepsilon$.

(18), (19) with estimates (a), (b) give the estimate

$$|\tau_i(u_0)| \leqslant M\,\varepsilon \exp(-\beta x_0/\sqrt{\varepsilon}) \text{ for } h^2 \geqslant \varepsilon$$

This, matrix estimate and (7) gives

$$(20) \qquad |u(x_0) - u_{(0)}| \leqslant M\,h\sqrt{\varepsilon} \text{ for } h^2 \geqslant \varepsilon$$

due to boundary layer function.

The proof and the estimates for the function w_0 is the similar. Now we must estimate the function $g(x)$.

Formal Taylor's expansion of $\tau_i(u) = R_h\,u_i - Q_h\,(L\bar{u}_i)$ has the form ([1]):

$$(21) \qquad \tau_i = \tau_i^{(0)}u + \tau_i^{(1)}u' + \tau_i^{(2)}u'' + \tau_i^{(3)}u''' + \ldots + R, \ i = 0(1)n$$

for $i = 0$, we have

$$\tau_0^{(0)} = \tau_0^{(1)} = 0 \text{ and } \tau_0^{(2)} = (\beta_1/h)\,\{h^2/2 - \varepsilon/p_0\,(1 - \rho_0 \operatorname{cth} \rho_0) - \varepsilon/p_1(\operatorname{sh}\rho_1/\rho_1 - 1)\}$$

When $h^2 \leqslant \varepsilon$ we obtain

$$(22) \qquad |\tau_0^{(2)}| \leqslant M\,h^2/\varepsilon$$

and for $h^2 \geqslant \varepsilon$ we have

$$(23) \qquad |\tau_0^{(2)}| \leqslant M\,(h + \sqrt{\varepsilon})$$

From (7), estimate of the matrix and (23), (22) we have

$$(24) \qquad |u(x_0) - u_{(0)}| \leqslant M\,h^2 \quad \text{for } h^2 \leqslant \varepsilon$$

and

$$(25) \qquad |u(x_0) - u_{(0)}| \leqslant M\,h\sqrt{\varepsilon} \quad \text{for } h^2 \geqslant \varepsilon$$

due to the function $g(x)$.

It is easy to check that $\tau_0^{(3)}$ and remainders are of the lower order than $\tau_0^{(2)}$.

Now we can formulate the main result:

THEOREM 1. *Let u_i, $i = 0(1)\,n$ be the approximation to the solution of problem (1) with mixed boundary conditions (2) — (3) with $\beta_2 = 0$. Let $p(x)$, $f(x)$ lie in $C^2[0,1]$ and $p(x) \geqslant \bar{p} > 0$ and $p'(0) = p''(1) = 0$. Then the following inequality:*

$$|u(x_i) - u_i| \leqslant Mh\min(h, \sqrt{\varepsilon})$$

holds. M is a constant independent of ε and h.

Proof. It follows from the estimate for the function u_0, w_0 and g, i.e. from (17), (20), (24), (25) and from the estimate of the function $w(x)$, and from (11).

The estimate for the last equation when $i = N$ is the similar. It is another case of boundary conditions, case (2 — 3) when $\beta_2 \neq 0$.

For variable parameter $q_i = \sqrt{p_i/\varepsilon}\,h$ where $p_i = p(x_i)$ the coefficients of (6) are

$$r_N^- = (\beta_2/h)(-\rho_{N-1}/\operatorname{sh}\rho_{N-1}),\ r_N^C = \alpha_2 + (\beta_2/h)\,(\rho_N \operatorname{cth} \rho_N),\ r_N^+ = 0$$

$$q_N^- = (\beta_2/h)\,(1/p_{N-1})(1 - \rho_{N-1}/\operatorname{sh}\rho_{N-1}),\ q_N^C = (\beta_2/h)(1/p_N)(\rho_N \operatorname{cth} \rho_{N-1})\ q_N^- = 0.$$

Then

$$\tau_N(u_0) = (\beta_2/h)\exp(-\sqrt{p_0/\varepsilon}\,x_N)\{-\rho_{N-1}/\operatorname{sh}\rho_{N-1}\exp(\rho_0) + \rho_0 + \rho_N \operatorname{cth}\rho_N - (p_0 - p_{N-1})/p_{N-1}\exp(\rho_0)\,(1 - \rho_{N-1}/\operatorname{sh}\rho_{N-1}) + (p_0 - \rho_N)/p_N\,(\rho_N \operatorname{cth} \rho_{N-1})\}$$

Developing those expressions at ρ_N we obtain:

$$(26) \qquad \tau_N(u_0) = (\beta_2/h)\exp(-\sqrt{p_0/\varepsilon}\,x_N)\left\{(\rho_0^2 - \rho_N^2)\left(\frac{1}{2} + \frac{\rho_0}{\sigma} + 0(\rho^2)\right) - (\rho_0^2 - \rho_N^2)\left(\frac{1}{2} + \frac{\rho_0}{\sigma} + 0(\rho^2)\right)\right\} + R$$

where

$$R = \exp(\rho_0)\,[-a(\rho_{N-1} - \rho_N) + (\rho_N - \rho_{N-1})/\rho(1 - \rho_N/\operatorname{sh}\rho_N) - a(\rho_N - \rho_{N-1})/\rho_N(\rho_{N-1} - \rho_N)] + 0(h^4/\varepsilon).$$

With a is denoted part in $\rho_{N-1}/\operatorname{sh}\rho_{N-1} = \rho_N/\operatorname{sh}\rho_N + (\rho_{N-1} - \rho_N)a$ where

$$a = \frac{\operatorname{sh}\rho_N - \rho_N \operatorname{ch}\rho_N}{sh^2\,\rho_N} = -\frac{\rho_N}{3}\left(1 - \frac{7}{30}\rho_N^2 + \frac{1}{30}\rho_N^4 + 0(\rho_N^\sigma)\right) \text{ for } h^2 \leqslant \varepsilon.$$

Then $R = (\beta_2/h)\exp(-\sqrt{\rho_0/\varepsilon}\,x_N)\,(\rho_{N-1} - \rho_N)\exp(\rho_0)\left(-\frac{\rho_N}{6} + 0(\rho_N^2)\right)$.

It gives

$$(27) \qquad |R| \leqslant M\,h^2/\varepsilon \qquad \text{for } h^2 \leqslant \varepsilon$$

It is obvious from (26) that main part of $|\tau_0(u_0)| \leqslant M\,h^3/\varepsilon$.

The estimate of R drops the order of uniform convergence to one. Exactly, (27) and estimate of matrix give

$$(28) \qquad |u(x_N) - u_N| \leqslant M\,h \text{ for } h^2 \leqslant \varepsilon.$$

Similarly as in proof of Theorem 1, we obtain when $h^2 \geqslant \varepsilon$

$$|u(x_N) - u_N| \leqslant M\, h \sqrt{\varepsilon}$$

In (21) estimate of $g(x)$ for $i = N$ gives : $\tau_N^{(0)} = \tau_N^{(1)} = 0$ and

$$\tau_N^{(2)} = \frac{h^2}{2}(r_{\bar N} - q_{\bar N}\, p_{N-1}) - \varepsilon(q_{\bar N} + q_N^c),$$

$$\tau_N^{(2)} = -\frac{h}{2}\beta_2 - \varepsilon\beta_2/h\, p_N\,(\rho_N + h\,\rho_{N/2} + N)$$

where N denotes a part which does not influence the order of uniform convergence.

When $h^2 \leqslant \varepsilon$, $\tau_N^{(3)} = -\beta_2/2 + \beta_2 h/2\left(1 + \frac{\rho_N^2}{12}\right) + N$

So we obtain $|\tau_N^{(2)}| \leqslant M\, h^3/\varepsilon$ for $h^2 \leqslant \varepsilon$ and

$$|\tau_N^{(2)}| \leqslant M(h + \sqrt{\varepsilon}) \text{ for } h^2 \geqslant \varepsilon.$$

It gives with matrix estimate and (7)

(29) $$|u(x_N) - u_N| \leqslant M\, h\sqrt{\varepsilon} \quad \text{for } h^2 \geqslant \varepsilon \text{ and}$$

$$|u(x_N) - u_N| \leqslant M\, h^2 \quad \text{for } h^2 \leqslant \varepsilon.$$

due to the function $g(x)$
Estimate of $w(x)$ is similar.
Above all implies

THEOREM 2. *Let condition of Theorem 1 be fulfilled where condition (2) and (3) are fulfiled. Then*

$$|u(x_i) - u_i| \leqslant \begin{cases} Mh & h^2 \leqslant \varepsilon \\ Mh\sqrt{\varepsilon} & h^2 \geqslant \varepsilon \end{cases}$$

M is a constant independent of ε and h, $i = 0(1)N$.

Proof. It follows from (11), Theorem 1, (28), (29) and the estimate of $w(x)$.

THEOREM 3. *If condition $p'(0) = p'(1) = 0$ in Theorem 2 is not satisfied then the following estimate*

(30) $$|u(x_i) - u_i| \leqslant M \min(h, \sqrt{\varepsilon}) \text{ holds}$$

Proof. When $h^2 \geqslant \varepsilon$, $|\tau_q| \leqslant M/h(p_0 - p_{N-1})\exp(-\sqrt{p_0/\varepsilon}\, x_{N-1})q_{\bar N}$
Since $|q_{\bar N}| \leqslant M$, $|p_0 - p_{N-1}| \leqslant M\, x_{N-1}$, using (a), (b) we obtain

(31) $$|\tau_q| \leqslant M\sqrt{\varepsilon}/h \exp(-\delta\, x_{N-1})/\varepsilon)$$

Table 1

ε \ k	1	2	3	4	5
1	0.95 0.134E+00	0.97 0.682E−01	0.99 0.344E−01	0.99 0.173E−01	1.00 0.866E−02
2^{-1}	0.95 0.210E+00	0.97 0.145E+00	0.99 0.528E−01	0.99 0.265E−01	1.00 0.133E−01
2^{-2}	0.94 0.310E+00	0.98 0.157E+00	0.98 0.789E−01	0.99 0.396E−01	1.00 0.199E−01
2^{-3}	0.93 0.453E+00	0.97 0.231E+00	0.99 0.117E+00	0.99 0.585E−01	1.00 0.293E−01
2^{-4}	0.91 0.659E+00	0.97 0.337E+00	0.99 0.170E+00	0.99 0.855E−01	1.00 0.428E−01
2^{-5}	0.88 0.946E+00	0.90 0.488E+00	0.98 0.247E+00	0.99 0.124E+00	1.00 0.621E−01
2^{-6}	0.81 0.134E+01	0.94 0.699E+00	0.98 0.355E+00	0.99 0.178E+00	1.00 0.894E−01
2^{-7}	0.67 0.185E+01	0.90 0.992E+00	0.97 0.507E+00	0.99 0.255E+00	1.00 0.128E+00
2^{-8}	0.43 0.246E+01	0.83 0.139E+01	0.95 0.720E+00	0.98 0.364E+00	1.00 0.183E+00
2^{-9}	— 0.308E+01	0.69 0.191E+01	0.91 0.102E+01	0.97 0.517E+00	0.92 0.260E+00
2^{-10}	— 0.345E+01	0.45 0.253E+01	0.84 0.142E+01	0.96 0.732E+00	0.99 0.369E+00
2^{-11}	— —	0.41 0.324E+01	0.70 0.194E+01	0.92 0.103E+01	0.98 0.523E+00

Table 2 gives a test for the same example with end condition of type (2—3), where $\beta_2 = 0$.

$$u(0) - u'(0) = 0, \; u(1) = 0$$

From the fact $\tau_r(\rho_0) = 0$ we obtain

(32) $$|\tau_r - \tau_r(\rho_0)| \leqslant M\sqrt{\varepsilon}/h \exp(-\delta\, x_{N-1})/\varepsilon)$$

(31), (32) and (b) yield

$$|\tau_N(u_0)| \leqslant M \exp(-\delta_1\, x_{N-1}/\varepsilon)$$

With matrix estimate it gives (30).
The estimate of the other parts is the same as in Theorem 2.

Table 2

ε \ k	1	2	3	4	5
1	2.00 0.484E−03	2.00 0.121E−03	2.00 0.303E−04	2.00 0.757E−04	2.00 0.189E−05
2^{-1}	2.00 0.685E−03	2.00 0.171E−03	2.00 0.428E−04	2.00 0.107E−04	2.00 0.268E−05
2^{-2}	2.00 0.817E−03	2.00 0.205E−03	2.00 0.512E−04	2.00 0.128E−04	2.00 0.320E−05
2^{-3}	1.99 0.861E−03	2.00 0.216E−03	2.00 0.539E−04	2.00 0.135E−04	2.00 0.337E−05
2^{-4}	1.97 0.897E−03	2.00 0.225E−03	2.00 0.563E−04	2.00 0.141E−04	2.00 0.352E−05
2^{-5}	1.92 0.122E−02	1.98 0.311E−03	2.00 0.778E−04	2.00 0.195E−04	2.00 0.487E−05
2^{-6}	1.91 0.183E−02	1.98 0.466E−03	1.99 0.117E−03	2.00 0.293E−04	2.00 0.733E−05
2^{-7}	1.76 0.255E−02	1.90 0.682E−03	1.99 0.172E−03	1.99 0.123E−04	2.00 0.108E−04
2^{-8}	1.18 0.372E−02	1.91 0.133E−02	1.98 0.359E−03	1.99 0.904E−04	2.00 0.228E−04
2^{-9}	0.29 0.461E−02	1.79 0.133E−02	1.89 0.359E−03	1.99 0.904E−04	1.99 0.227E−04
2^{-10}	— —	1.23 0.193E−02	1.92 0.511E−03	1.98 0.130E−03	1.99 0.326E−04
2^{-11}	— —	0.41 0.238E−02	1.81 0.680E−03	1.89 0.184E−03	1.99 0.463E−04

Remark: In this example the condition in Theorems $p'(0) = p'(1) = 0$ is not satisfied.

Numerical Experiments. The scheme (4) — (6) is tested on the example

$$-\varepsilon u'' + (1+x)^2 u = (4x^2 - 14x + 4)(1 + x^2)$$

with conditions $u(0) - u'(0)$, $u(1) + u'(1) = 0$ taken from [2].

The program was written in Fortran IV plus and executed on Delta 340 (PDP 11/34) in double precision mode with 16 significant figures. The technique which is used here is described in [3]. There this scheme is tested on several examples with mixed boundary conditions.

Table 1 shows observed rates of uniform convergence obtained over uniform mesh. The second line in each row is the difference between two consecutive meshes (for $1/N$ and $1/2N$) given in max norm.

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