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CONVEXITY AND FUNCTIONALS

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1. **Convexity.** There are many generalizations of the convexity of real functions. Some of them are surveyed in [7]. Let us recall here those which we use in what follows. We denote by:

$$C[a, b] = \{f : [a, b] \to \mathbb{R}, \text{ continuous}\}$$

$$K[a, b] = \{f \in C[a, b];\ f(tx + (1 - t)y) \leqslant tf(x) + (1 - t)f(y),$$
$$\forall\, t \in I,\ \forall x, y \in [a, b]\}, \text{ where } I = [0, 1]$$

$$Q[a, b] = \{f \in C[a, b];\ f(tx + (1 - t)y \leqslant \max\,(f(x), f(y)),$$
$$\forall\, y, x \in [a, b],\ \forall\, t \in I\}$$

$$C(b) = \{f \in C[0, b],\ f(0) = 0\}$$

$$K_p(b) = \{f \in C(b);\ f(tx + p(1 - t)y) \leqslant tf(x) + p(1 - t)f(y),\ \forall\, t \in I,$$
$$\forall\, x, y \in [0, b]\} \text{ for } p \in I$$

$$K_J(b) = \{f \in C(b);\ f(tx + sy) \leqslant tf(x) + sf(y),\ \forall\,(t, s) \in J,$$
$$\forall\, x, y \in [0, b]\}, \text{ with } J \subset I \times I$$

$$S^*(b) = K_0(b)$$

$$S(b) = \{f \in C(b);\ f(x + y) \geqslant f(x) + f(y),\ \forall\, x, y, x + y \in (0, b]\}$$

the sets of continuous, convex, quasi-convex functions on $[a, b]$, respectively continuous, p-convex, J-convex, starshaped, superadditive functions on $[0, b]$ with $f(0) = 0$.

Taking into account all these classes of functions, we are led naturally to the following general definition.

Let L be a set of functionals defined on a set M of functions.

Definition 1. A function $f \in M$ is said to be convex with respect to the set L (or L-convex) if:

$$A(f) \geqslant 0,\quad \forall\, A \in L.$$

We denote by L^+M the set of L-convex functions from M.

Remark 1. A similar definition is given in [2] and [3] for the elements of a vector space but having in view other problems.

It is easy to indicate the sets of functionals which define each of the above classes. We use mainly the functional of evaluation, given by:

$$E_x(f) = f(x).$$

Then, the sets of convex, p-convex, J-convex, starshaped, superadditive and quasi-convex functions are defined respectively by the sets of functionals:

$$K = \{tE_x + (1 - t)E_y - E_{tx+(1-t)y}\,;\ t \in I,\ x, y \in [a, b]\}$$

$$K_p = \{tE_x + p(1 - t)E_y - E_{tx+p(1-t)y}\,;\ t \in I,\ x, y \in [0, b]\}$$

$$K_J = \{tE_x + sE_y - E_{tx+sy}\,;\ (t, s) \in J,\ x, y \in [0, b]\}$$

$$S^* = \{tE_x - E_{tx}\,;\ t \in I,\ x \in [0, b]\}$$

$$S = \{E_{x+y} - E_x - E_y\,;\ x, y, x + y \in [0, b]\}$$

$$Q = \{\max(E_x, E_y) - E_{tx+(1-t)y}\,;\ t \in I,\ x, y \in [a, b]\}.$$

We have thus:

$$K^+C[a, b] = K[a, b],\ K_p^+C(b) = K_p(b), \text{ and so on.}$$

2. **Inequalities.** As it is known for $K[a, b]$ (see [14] and [10]), the set of functionals:

$$L^+ = \{A : M \to \mathbb{R}\,;\ A(f) \geq 0,\ \forall f \in L^+M\}$$

may be generally much more rich than L.

What we can say in general about it? We shall indicate three ways for construction of elements from L^+.

If we consider the convex conical span of the set L:

$$\text{cone}\ (L) = \{A : M \to \mathbb{R}\,;\ \exists n \in N,\ \exists t_1, \ldots, t_n \geq 0,\ \exists A_1, \ldots, A_n \in L,\ A = t_1A_1 + \ldots + t_nA_n\}$$

we have easily the following:

LEMMA 1. *For every set of functionals L, holds the inclusion:*

$$\text{cone}\ (L) \subset L^+.$$

Also we can consider a generalized adherence of cone (L) by:

$$\text{clcone}\ (L) = \{A : M \to \mathbb{R}\,;\ \forall n \in N,\ \exists A_n \in \text{cone}\ (L),\ \forall f \in M\ A(f) = \liminf_{n\to\infty} A_n(f)\}.$$

Then we have also:

LEMMA 2. *For every set L, holds:*

$$\text{clcone}\ (L) \subseteq L^+.$$

DEFINITION 2. Two sets of functionals L and L' are in relation $L' \geq L$ if for every $B \in L'$ there is an $A \in L$ such that $B(f) \geq A(f)$ for every $f \in M$.

LEMMA 3. *If $L' \geq L$ then $L' \subset L^+$.*

There are many papers which prove inequalities for convex functions. In fact, they establish the belonging of some functional to the corresponding set L^+. We want to analyse some of them from this point of view.

We begin with the most familiar of them, the inequality of Jensen: if f is a convex function on $[a, b]$, $x_1, \ldots, x_n \in [a, b]$ and $c_1, \ldots, c_n$ are positive constants, then:

$$f\left(\sum_{k=1}^{n} c_kx_k \Big/ \sum_{k=1}^{n} c_k\right) \leq \sum_{k=1}^{n} c_kf(x_k) \Big/ \sum_{k=1}^{n} c_k.$$

This is equivalent with the belonging to K^+ of the functional:

$$J_n = \sum_{k=1}^{n} c_kE_{x_k} \Big/ \sum_{k=1}^{n} c_k - E_{X_n}$$

where we have denoted:

$$X_m = \sum_{k=1}^{m} c_kx_k \Big/ \sum_{k=1}^{m} c_k.$$

Writing J_n as:

$$J_n = \sum_{k=2}^{n} [(c_1 + \ldots + c_{k-1})\ E_{X_{k-1}} + c_k E_{x_k} - (c_1 + \ldots + c_k)\ E_{X_k}] \Big/ \sum_{i=1}^{n} c_i$$

it results that:

$$J_n \in \text{cone}\ (K).$$

Another well-known inequality is that of Hadamard: for $f \in K[a, b]$ hold the relations:

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a}\int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2}.$$

To prove it we show that:

$$(E_a + E_b)/2 - M_{a,b},\quad M_{a,b} - E_{(a+b)/2} \in \text{clcone}\ (K)$$

where we have denoted by $M_{a,b}$ the functional defined by:

$$M_{a,b}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

But $(E_a + E_b)/2 - M_{a,b}$ is the (punctual) limit of the following sequence of functionals :

$$\frac{1}{2} E_a + \frac{1}{2} E_b - \frac{1}{n} \sum_{i=1}^{n} E_{a+(i-1/2)(b-a)/n} =$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{n-i+1/2}{n} E_a + \frac{i-1/2}{n} E_b - E_{\frac{n-i+1/2}{n} a + \frac{i-1/2}{n} b} \right]$$

while $M_{a,b} - E_{(a+b)/2}$ may be obtained as the limit of the sequence :

$$\frac{1}{2n} \left(\sum_{i=0}^{n-1} E_{a+ih} + E_{b-ih} \right) - E_{(a+b)/2} =$$

$$= \frac{1}{2n} \sum_{i=0}^{n-1} (E_{a+ih} + E_{b-ih} - 2E_{(a+b)/2}),$$

where $h = (b - a)/2n$. As these are elements of cone(K), the results follow.

3. Hierarchies of convexity. On this way we can also easily compare two convexity classes.

LEMMA 4. *We have $L_1^+ M \subset L_2^+ M$ if and only if $L_2 \subset L_1^+$.*

Remark 2. Lemmas 1 – 3 offer methods for obtaining such subsets L_2.

DEFINITION 3. For the two subsets J and H of $I \times I$, the relation $J \geqslant H$ means that for any $(t, s) \in H$ there is an $(t, r) \in J$ with $s \leqslant r$. Also $H \geqslant 0$ means that $H \geqslant I \times \{0\}$.

THEOREM 1. *Hold the following relations :*

(a) $S^* \subset K$ *on* $C(b)$

(b) $S^* \subset K_p$ *on* $C(b)$

(c) $S^* \subset K_J$ *on* $C(b)$ *if* $J \geqslant 0$

(d) $Q \geqslant K$ *in* $C[a, b]$

(e) *if* $p \leqslant q$ *then* $K_p \subset K_q$ *on* $C(b)$

(f) *if* $J \geqslant H \geqslant 0$ *then* $K_H \subset K_J$ *on* $C(b)$

(g) $S \subset$ cone(S^*) *on* $C(b)$.

Proof. (a), (b) and (c) follow by taking $y = 0$.

(d) We have for any $f \in C[a, b]$:

$$\max\,(E_x(f), E_y(f)) - E_{tx+(1-t)y}(f) = [t + (1 - t)] \max\,(E_x(f), E_y(f)) - E_{tx+(1-t)y}(f) \geqslant tE_x(f) + (1 - t)\,E_y(f) - E_{tx+(1-t)y}(f)$$

(e) For $p \leqslant q$, it follows :

$$tE_x + p(1 - t)E_y - E_{tx+(1-t)y} =$$

$$= tE_x + (1 - t)E_{(p/q)y} - E_{tx+q(1-t)(p/q)y} + q(1 - t)\,((p/q)E_y - E_{(p/q)y})$$

that is, it belongs to cone(K_q) because $S^* \subset K_q$ by (b).

(f) If $J \geqslant H$, then for $(t, s) \in H$ there is an $(t, r) \in J$ with $r \geqslant s$. So :

$$tE_x + sE_y - E_{tx+sy} = tE_x + rE_{(s/r)y} - E_{tx+r(s/r)y} + r\left(\frac{s}{r} E_y - E_{(s/r)y}\right)$$

and the conclusion follows because $S^* \subset K_J$.

(g) If $x, y, x + y \in [0, b]$:

$$E_{x+y} - E_x - E_y = \frac{x}{x+y} E_{x+y} - E_x + \frac{y}{x+y} E_{x+y} - E_y =$$

$$= \left[\frac{x}{x+y} E_{x+y} - E_{\frac{x}{x+y}(x+y)}\right] + \left[\frac{y}{x+y} E_{x+y} - E_{\frac{y}{x+y}(x+y)}\right].$$

These relations give the following known results :

THEOREM 2. *Hold the following inclusions :*

$K_1(b) = K(b) \subset S^*(b) \subset S(b)$, (see [1] and [8])

$K_q(b) \subset K_p(b)$ *if* $q \geqslant p$, (see [7] or [11])

$K_J(b) \subset K_H(b)$ *if* $J \geqslant H \geqslant 0$ (see [7] or [9])

$K[a, b] \subset Q[a, b]$.

In [12] we have generalized the first chain of inclusions for convexity of higher order. Let us remind that the divided differences (on the distinct points $x_0, x_1, \ldots$) are defined recurrently by :

$$[x_0 ; f] = f(x_0),\ [x_0, \ldots, x_n ; f] = ([x_0, \ldots, x_{n-1} ; f] - [x_1, \ldots, x_n ; f])/(x_0 - x_n).$$

One considers the set of functions convex of order n :

$$K_n[a, b] = \{f : [a, b] \to \mathrm{R},\ [x_0, \ldots, x_n ; f] \geqslant 0,\ \forall\, x_0, \ldots, x_n \in [a, b] \text{ distinct}\}$$

the set of starshaped of order n functions :

$$S_n^*[a, b] = \{f : [a, b] \to \mathbb{R},\ [a, x_1, \ldots, x_n ; f] \geqslant 0\ \ \forall x_1, \ldots, x_n \in [a, b] \text{ distinct}\}$$

and that of functions superadditive of order n :

$$S_n[a, b] = \{f : [a, b] \to \mathbb{R},\ \forall x_1, \ldots, x_n \in (a, b] \text{ distinct}$$

and $x_1 + \ldots + x_n - na \leqslant b - a$ implies $\sum_{k=0}^{n} (-1)^{n-k} \cdot \sum_{(k)} f(x_{i_1} + \ldots + x_{i_k} - (k-1)a) \geqslant 0\}$

where $\sum_{(k)}$ means the sum over all the combinations of indices for $k>0$ and $f(a)$ for $k = 0$.

These sets of functions are defined by the sets of functionals:

$$K_n = \{[x_0, x_1, \ldots, x_n; \cdot];\ x_0, x_1, \ldots, x_n \in [a, b] \text{ distinct}\}$$

$$S_n^* = \{[a, x_1, \ldots, x_n;\ \cdot];\ x_1, \ldots, x_n \in (a, b] \text{ distinct}\}$$

respectively:

$$S_n = \left\{\sum_{k=0}^{n} (-1)^{n-k} \sum_{(k)} E_{x_{i_1} + \ldots + x_{i_k} - (k-1)a};\right.$$

$$\left. x_1, \ldots, x_n \in [a, b] \text{ distinct},\ x_1 + \ldots + x_n - na \leqslant b - a\right\}.$$

Obviously:

$$S_n^* \subset K_n,\ \forall n.$$

In [12] we have proved that:

$$E_{x+y-a} - E_x - E_y + E_a = (x-a)(y-a)([a, x, x+y-a; \cdot] + [a, y, x+y-a; \cdot])$$

$$E_{x+y+z-2a} - E_{x+y-a} - E_{x+z-a} - E_{y+z-a} + E_x + E_y + E_z - E_a =$$
$$= (x-a)(y-a)(z-a)([a, x, x+y-a, x+z-a; \cdot] +$$
$$+ [a, y, x+y-a, y+z-a; \cdot] + [a, z, x+z-a, y+z-a; \cdot] +$$
$$+ [a, x+y-a, x+z-a, x+y+z-2a; \cdot] +$$
$$+ [a, x+y-a, y+z-a, x+y+z-2a; \cdot] +$$
$$+ [a, x+z-a, y+z-a, x+y+z-2a; \cdot])$$

and analogously every functional from S_4 can be expressed as a sum of $4!$ functionals from S_n^*. Thus:

$$S_n \subset \operatorname{cone}(S_n^*) \text{ for } n = 2,3,4.$$

So we have proved:

THEOREM 3. *For every interval $[a, b]$ hold the inclusions*:

$$K_n[a, b] \subset S_n^*[a, b] \subset S_n[a, b], \textit{for } n = 2,3,4.$$

These inclusions generalize the relations:

$$K_n[a, b] \subset S_n[a, b]$$

proved for $n = 2$ by M. Petrović in [5], for $n = 3$ by P. M. Vasić in [13] and for $n = 4$ by J. D. Kečkić in [4]. So they also generalize the corresponding results of T. Popoviciu from [6].

Taking into account these results, we can remark that in some cases an inequality is valid in more general conditions than those in which it is given. So are those form [5], [13] and [4]. We remind here that from [4] which generalizes all of them: if $f \in K_n[0, b]$, then for every $m \geqslant n$, $x_i \in [0, b]$, $i = 1, \ldots, m$, $x_1 + \ldots + x_m \leqslant b$, holds:

$$f(x_1 + \ldots + x_m) + \sum_{k=0}^{n-1} (-1)^{n-k} \binom{m-k-1}{n-k-1} \sum_{(k)} f(x_{i_1} + \ldots + x_{i_k}) \geqslant 0.$$

In fact this is valid for every function f from $S_n[0, b]$ and it follows even from the original proof.

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