MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION,
Tome 19, N° 1, 1990, pp. 1–6

THE ANGLE BETWEEN TWO SETS AND THEIR INNER PRODUCT IN ANY BANACH SPACE

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Abstract. Based on observation (12) concerning the definition of sin 0 in the Euclidean plane geometry, a definition of the angle between two sets and their inner product in any Banach space, or for that matter, in any metric (or even in a much more abstract) space is introduced.

Our motivation is based on the following considerations in connection with sin θ, where θ is the angle between the bounded line segment A and the line segment B (in the usual Euclidean plane) intersecting at the point 0 as shown in Fig. 1.

(1) The line segment A is a set of points each denoted by $+$

(2) The line segment B is a set of points each denoted by.

(3) The line segments A and B have a nonempty intersection, namely $\{0\}$.

(4) Let $d(p, q)$ denote the Euclidean distance of the point p from the point q in the Euclidean plane.

5) In Fig. 1, let b be the foot of the perpendicular from the point a to the line segment B. Then the Euclidean distance $d(a, B)$ of the point a from the line segment B (using notation in (4)) is obviously $d(a, b)$, i.e.,

6) $d(a, B) = d(a, b) = \min\{d(a, x) : x \in B\}$

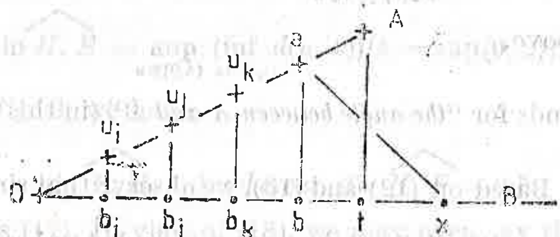

Fig. 1

(7) As Fig. 1 shows, the bounded line segment A is the closed interval $[0, r]$ of length $r > 0$.

(8) In Fig. 1, for every point $u_i \in [0, r]$, let $b_i \in B$ denote the foot of the perpendicular from u_i to the line segment B. Moreover, let t denote the

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1980 *Mathematics Subject Classification.* Primary 46B20 foot of the perpendicular from the point r of A to the line segment B. Then clearly (using notation in (4) and (5)) we have

$$(9) \qquad |\overline{Or}| \sin \theta = d(r, t) = \max \{d(u_i, B) : u_i \in [0, r]\}$$

$$= \max \{d(u_i, b_i) : u_i \in [0, r]\}$$

However, from (6) it follows that

$$d(u_i, B) = \min \{d(u_i, x) : x \in B\}$$

which by (9) implies

$$(10) \qquad \sin \theta = (\max \{\min \{d(u_i, x) : x \in B\} : u_i \in [0, r]\})/r$$

Using a more familiar notation, (10) can be expressed as

$$(11) \qquad \sin \theta = \max_{u \in [0,r]} (\min_{x \in B} d(u, x))/r$$

where the subscript i is droppped from u_i since it was introduced in (8) only for the convenience of indicating that b_i is the foot of the perpendicular eminating from u_i.

Clearly, (11) can be more propitiously rewritten as

$$(12) \qquad \sin \widehat{A, B} = \sup_{u \in [0,r]} (\inf_{x \in B} d(u, x)) / r \textit{ with } r = \sup_{x \in [0,r]} d(x, 0)$$

From (12), with r as in the above, we deduce

$$(13) \qquad \widehat{A, B} = \arcsin (\sup_{u \in [0,r]} (\inf_{x \in B} d(u, x))/r \textit{ with } 0 \leqslant \widehat{A, B} \leqslant \frac{\pi}{2}$$

where in the above,

(14) $\widehat{A, B}$ stands for "*the angle between A and B*" (in this order)

Remark 1. Based on (12) and (13), we observe that $\sin \widehat{E, H}$ as well as $\widehat{E, H}$ can be readily defined for every bounded subset E and every subset H of the Euclidean plane which (for the sake of simplicity) have a unique point, say, 0, in common. Let us call the real number r given by

(15) $r = \sup_{x \in E} d(x, 0)$ the "central radius of E" (w.r.t.0)

As an example, let E, in Fig. 2, be the bounded set of points (each denoted by $+$) in the plane and H be a set of points (each denotd by.) with 0 as their unique common point. As shown in Fig. 2, every point of the set E

is on or inside the circle of radius 2. Hence, the central radius of E (w.r.t.0) is 2. Thus, using notation (14), in view of (12), we have

$$(16) \qquad \sin \widehat{E, H} = \sup_{u \in E} (\inf_{x \in H} d(u, x))/2$$

since $r = \sup_{x \in E} d(x, 0) = 2$.

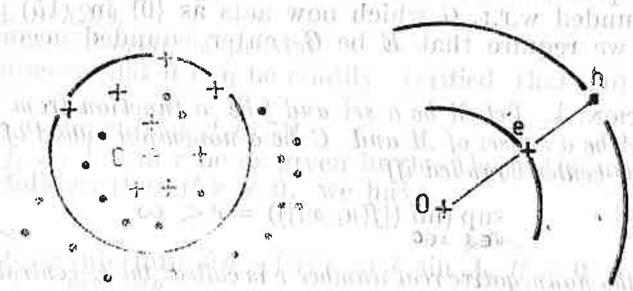

Fig. 2 Fig. 3

In general, for two bounded sets E and H in the plane

$$(17) \qquad \sin \widehat{E, H} \neq \sin \widehat{H, E} \textit{ and therefore } \widehat{E, H} \neq \widehat{H, E}$$

For instance, in Fig. 3, let $E = \{0, e\}$ and $H = \{0, h\}$ with central radii respectively equal to 2 and 4. Then, according to (16), we have

$$\sin \widehat{E, H} = \sup_{u \in \{0,e\}} (\inf_{x \in \{0,h\}} d(u, x))/2 = (\sup\{0, 2\})/2,$$

whereas

$$\sin \widehat{H, E} = \text{aup}_{u \in \{0,h\}} (\inf_{x \in \{0,e\}} d(u, x))/4 = \sup\{0, 2\})/4.$$

Therefore, in Fig. 3, we have

$$(18) \qquad \sin \widehat{E, H} = 1 \textit{ whereas } \sin \widehat{H, E} = 0.5$$

which validates (17). In view of (18), we may even say that in Fig. 3

the set $\{0, e\}$ *is orthogonal to the set* $\{0, h\}$

whereas

the angle between the set $\{0, h\}$ *and the set* $\{0, e\}$ is $\pi/6$.

Remark 2. The above examples pertain to sets in the Euclidean plane where $d(u, x)$ in (16) refers to the Euclidean distance between the points u and x of the plane.

Let us observe however, that (16) may be made meaningful in connection with points u and x of any (abstrct) set M and a function f from the Cartesian product $M \times M$ into the set of, say, real numbers. Moreover, this can be done without requiring that f satisfy any of the specific properties of a distance or a metric function. Furthermore, this can be done without requiring that E and H in (16) have a unique point in common. All that we require is that E and H have a nonempty intersection G and that E be bounded w.r.t. G which now acts as $\{0\}$ in (15) and (16). In other words, we require that E be G-center bounded according to the following :

DEFINITION. 1. *Let M be a set and f be a function from $M \times M$ into the reals. Let A be a subset of M and C be a nonempty subset of M. Then we say that A is C-center bounded iff*

(19) $$\sup_{u \in A}(\inf_{x \in C}(|f(u, x)|)) = r < \infty$$

in which case the nonnegative real number r is called the C-central radius of A.

Next, based on the above and motivated by (12) and (13), we introduce the following :

DEFINITION 2. *Let M be a set and f be a function from $M \times M$ into the reals. Let A and B be subsets of M with a nonempty intersection C and let A have a finite C-central radius r given by (19). Then the real number $\widehat{A, B}$ given below is called the angle between the set A and the set B (in this order)*

(20) $$\widehat{A, B} = \arcsin\left(\sup_{u \in A}(\inf_{x \in B}(|f(u, x)|)/r)\right) \text{ with } 0 \leqslant \widehat{A, B} \leqslant \frac{\pi}{2}$$

if $r \neq 0$, and $\widehat{A, B} = 0$ if $r = 0$.

Let us observe that if the sup appearing in (20) exists, then it is a nonnegative real number. Thus, to justify our Definition, we must prove that the sup appearing in (20) exists and its value is $\leqslant r$. This is shown in

THEOREM. *Let M, f, A, B, C and r be as in Definition 2. Then the following sup exists and*

(21) $$0 \leqslant \sup_{u \in A}(\inf_{x \in B}(|f(u, x)|)) \leqslant r$$

Proof. For every $u \in A$, let us consider the set

(22) $$L_u = \{|f(u, x)| : x \in B\} \text{ with } u \in A$$

Since by the hypothesis $C \subseteq B$ and C is nonempty, from (22) it follows that for every $u \in A$ it is the case that L_u is a nonempty set of nonnegative real numbers and therefore $\inf L_u$ exists. However, since $C \subseteq B$, we have

(23) $$0 \leqslant \inf L_u \leqslant \inf_{x \in C}(|f(x, u)|) \text{ for every } u \in A$$

Next, let us consider the set

(24) $$H = \{\inf L_u : u \in A\}$$

Again, since by the hypothesis $C \subseteq A$ and C is nonempty, it follows that H is a nonempty set of real numbers. Moreover, from (19) and (23) we see that $\sup H \leqslant \sup_{u \in A}(\inf_{x \in C}(|f(u, x)|) = r$. But $\sup H$ is precisely the sup appearing in (21).

Hence, the Theorem is proved.

Let us observe that it can be readily verified that (20) implies that $\widehat{A, A} = 0$ for every subset A of M.

Let M, f, A, B and r be as given in the above Definition 2. Then from (20) it follows that if $r \neq 0$, we have

(25) $$\sin\widehat{A, B} = \sup_{u \in A}(\inf_{x \in B}(|f(u, x)|))/r \text{ and } \sin\widehat{A, B} = 0 \text{ if } r = 0$$

Based on (25), as expected, we define $\cos\widehat{A, B}$ as follows

(26) $$\cos\widehat{A, B} = +(1 - \sin^2\widehat{A, B})^{1/2}$$

For a subset S of M, let as usual *diam*(S) stand for the "diameter *of* S", where

(27) $$\operatorname{diam}(S) = \sup\{|f(x, y)| : (x, y) \in S \times S\}$$

Finally, based on (26) and (27), we define the *"inner product"* $A \cdot B$ of the set A and the set B (in this order) to be the real number given by

(28) $$A \cdot B = \operatorname{diam}(A) \cdot \operatorname{diam}(B) \cdot \cos\widehat{A, B} \text{ with } \operatorname{diam}(A) \cdot \operatorname{dim}(B) < \infty$$

Remark 3. In the above Definiton, $\widehat{A, B}$ is defined for the case where the C-central radius r of A is bounded, *i.e.*, $r < \infty$. The case $r = \infty$, as usual is handled as a limit of bounded cases.

Remark 4. Let us observe that in Fig. 1, the angle θ gives a measure of the wedge between the lines (sets) A and B. Analogously, $\widehat{A, B}$ as introduced by (20) may be used to give a measure of the wedge or the "gap" between two subsets A and B of an abstract space M with respect to a function f mapping $M \times M$ into the set of, say, real numbers. Similarly, the inner produce $A \cdot B$ as introduced by (28) may be used in defining the notion of the "projection" of a set A on a set B. Even if M and f are quite untamed, the concepts of $\widehat{A, B}$ and $A \cdot B$ may still prove to be useful.

Clearly, the concepts $\widehat{A, B}$ and $A \cdot B$ as introduced in (20) and (28), may become more useful if M shares more properties with a Euclidean space and f shares more properties with a metric function. In fact, this would be the case if A and B are subsets (with a nonempty intersection) of a Banach space M and f is the metric derived from the norm of M. See the References below.

The author thanks Dr. Tryphon T. Georgiou for a brief but helpful discussion.

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Received 7.XI.1989

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