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SOME REMARKS ON ČEBYŠEV'S INEQUALITY

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Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ and $p = (p_1, \ldots, p_n)$ be n-tuples of real numbers and denote :

$$T_n(a, b, p) = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i.$$

It is well known that if $p_i \geqslant 0$ $(i = \overline{1, n})$ and a, b are similarly ordered, i.e., $(a_i - a_j)(b_i - b_j) \geqslant 0$ for all $i, j = 1, \ldots, n$, then the following inequality holds :

$$T_n(a, b, p) \geqslant 0. \tag{1}$$

In the sequel we shall give an improvement of this fact. For other results which contain refinements of Čebyšev's inequality see [1 — 5] where further references are given.

THEOREM 1. *Let a, b be similarly ordered n-tuples and p be an n-tuple of real numbers. If we denote $|p| = (|p_1|, \ldots, |p_n|)$ then the following inequality holds :*

$$T_n(a, b, |p|) \geqslant |T_n(a, b, p)| \geqslant 0. \tag{2}$$

Proof. We shall use the following simple identity :

$$T_n(a, b, x) = \frac{1}{2} \sum_{i,z=1}^{n} [x_i x_j (a_i - a_j)(b_i - b_j)$$

Then we have :

$$|T_n(a, b, p)| \leqslant \frac{1}{2} \sum_{i,j=1}^{n} |p_i p_j| \, |(a_i - a_j)(b_i - b_j)| =$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} |p_i| \, |p_j| \, (a_i - a_j)(b_i - b_j) = T_n(a, b, |p|)$$

and the statement is proven.

COROLLARY 1. *Let a, b be as above and $e_i \in \{-1, 1\}$ ($i = \overline{1, n}$). Then the following inequality is true:*

$$(2)\quad n\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \geqslant \left| \sum_{i=1}^{n} e_i \sum_{i=1}^{n} e_ia_ib_i - \sum_{i=1}^{n} e_ia_i \sum_{i=1}^{n} e_ib_i \right| \geqslant 0.$$

Remarks 1°. If $n = 2k$ and $e_i = (-1)^i$ the inequality (2) becomes:

$$(3)\quad 2k\sum_{i=1}^{2k} a_ib_i - \sum_{i=1}^{2k} a_i \sum_{i=1}^{2k} b_i \geqslant \left| \sum_{i=1}^{2k} (-1)^i a_i \sum_{i=1}^{2k} (-1)^i b_i \right| \geqslant 0.$$

2°. If $n = 2k + 1$ then we have:

$$(4)\quad (2k+1)\sum_{i=1}^{2k+1} a_ib_i - \sum_{i=1}^{2k+1} a_i \sum_{i=1}^{2k+1} b_i \geqslant \left| \sum_{i=1}^{2k+1} (-1)^i a_ib_i + \sum_{i=1}^{2k+1} (-1)^i a_i \sum_{i=1}^{2k+1} (-1)^i b_i \right| \geqslant 0.$$

COROLLARY 2. *Let a be an n-tuple of real numbers and e_i as above ($i = \overline{1, n}$). Then the following inequality holds:*

$$(5)\quad n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 \geqslant \left| \sum_{i=1}^{n} e_i \sum_{i=1}^{n} e_i a_i^2 - \left(\sum_{i=1}^{n} e_i a_i\right)^2 \right| \geqslant 0.$$

Remarks 1°. If $n = 2k$, by the above inequality we have:

$$(6)\quad 2k\sum_{i=1}^{2k} a_i^2 - \left(\sum_{i=1}^{2k} a_i\right)^2 \geqslant \left(\sum_{i=1}^{2k} (-1)^i a_i\right)^2 \geqslant 0.$$

2. If $n = 2k + 1$, then (5) becomes for $e_i = (-1)^i$ ($i = \overline{1, n}$)

$$(2k+1)\sum_{i=1}^{2k+1} a_i^2 - \left(\sum_{i=1}^{2k+1} a_i\right)^2 \geqslant \left| \sum_{i=1}^{2k+1} (-1)^i a_i^2 + \left(\sum_{i=1}^{2k+1} (-1)^i a_i\right)^2 \right| \geqslant 0.$$

(7)

Now, we shall give the integral analogous of Theorem 1.

Let f, g be continuous on $[a, b]$ and p be an integrable function on $[a, b]$. Denote:

$$T(f, g, p) = \int_a^b p(x)\mathrm{d}x \int_a^b p(x)f(x)g(x)\,\mathrm{d}x - \int_a^b p(x)f(x)\,\mathrm{d}x \int_a^b p(x)g(x)\mathrm{d}x.$$

It is well known that if f, g are similarly ordered, i.e., $(f(x) - f(y))(g(x) - g(y)) \geqslant 0$ for all $x, y \in [a, b]$ and p is non negative then the following integral inequality holds:

$$(8)\quad T(f, g, p) \geqslant 0.$$

Further on, we shall give an improvement of this fact.

THEOREM 2. *Let f, g be continuous and similarly ordered and p be an integrable function on $[a, b]$. Then the following inequality holds:*

$$(9)\quad T(f, g, |p|) \geqslant |T(f, g, p)| \geqslant 0$$

Proof. We start to the following integral identity:

$$T(f, g, h) = \frac{1}{2}\int_a^b\int_a^b h(x)h(y)(f(x) - f(y))(g(x) - g(y))\mathrm{d}x\mathrm{d}y.$$

Now, the proof follows by an argument similar to that in the proof of the above theorem and we omit the details.

COROLLARY. *Let f be continuous on $[a, b]$ and p be integrable on $[a, b]$. Then the next integral inequality is valid:*

$$(10)\quad \int_a^b p(x)\,\mathrm{d}x \int_a^b p(x)\,f^2(x)\mathrm{d}x - \left(\int_a^b p(x)f(x)\mathrm{d}x\right)^2 \geqslant \left| \int_a^b p(x)\mathrm{d}x \int_a^b p(x)f^2(x)\,\mathrm{d}x - \left(\int_a^b p(x)f(x)\mathrm{d}x\right)^2 \right|$$

Now, if we return to the discrete case, we must recal a well-known refinement of Čebyšev's inequality (see for example [5]). If a, b are two similarly ordered n-tuples of real numbers and $p_i \geqslant 0$ ($i = \overline{1, n}$) then:

$$(11)\quad T_n(a, b, p) \geqslant T_{n-1}(a, b, p) \geqslant \ldots \geqslant T_2(a, b, p) \geqslant 0.$$

We note that the following result is also valid.

THEOREM 3. *Let a, b be as above and p be an n-tuple of real numbers. Then the following inequality holds:*

$$(12)\quad T_n(a, b, |p|) - T_{n-1}(a, b, |p|) \geqslant \geqslant |T_n(a, b, p) - T_{n-1}(a, b, p)| \geqslant 0.$$

Proof. It is easy to see that for all q a n-tuple of real numbers we have:

$$T_n(a,\ b,\ q) - T_{n-1}(a,\ b,\ q) = \sum_{i=1}^{n} q_i q_n (a_i - a_n)(b_i - b_n).$$

Then we have:

$$|\,T_n(a,\ b,\ p) - T_{n-1}(a,\ b,\ p)\,| \leq \sum_{i=1}^{n-1} |\,p_i p_n\,|\,|(a_i - a_n)(b_i - b_n)\,| =$$

$$\sum_{i=1}^{n-1} p_i p_n (a_i - a_n)(b_i - b_n) = T_n(a,\ b,\ |\,p\,|) - T_{n-1}(a,\ b,\ |\,p\,|)$$

and the statement is proven.

The corollaries of Theorem 1 can be reformuled also in this case, but we omit the details.

Now, as in [5] we shall consider the following mapping:

$$C_n(a,\ p) = \sum_{i,j=1}^{n} p_{ij}\, a_{jj} - \sum_{i,j=1}^{n} p_{ij} a_{ij},$$

where a_{ij} and p_{ij} $(1 \leq i,\ j \leq n)$ are real numbers. For some inequalities containing $C\,(a,\ b)$ which generalize or improve Čebyšev's inequality see [5].

We note that the following theorem holds.

THEOREM 4. *Let a_{ij} be real numbers such that:*

(13) $$a_{ii} + a_{jj} \geq a_{ij} + a_{ji} \text{ for all } i,\ j = 1,\ \ldots,\ n$$

Then for all p_{ij} real numbers such that $p_{ij} = p_{ji}$ $(1 \leq i,\ j \leq n)$ the following inequality is valid:

(14) $$C_n(a,\ |\,p\,|) \geq |\,C_n(a,\ p)\,| \geq 0.$$

Proof. We start to the following identity which holds for every x such that $x_{ij} = x_{ji}$ for all $i,\ j = 1,\ \ldots,\ n$.

$$C_n\,(a,\ x) = \frac{1}{2}\sum_{i,j=1}^{n} x_{ij}(a_{ii} + a_{ij} - a_{ij} - a_{jj})$$

Then we have

$$|\,C_n\,(a,\ p)\,| \leq \frac{1}{2}\sum_{i,j=1}^{n} |\,p_{ij}\,|\ |a_{jj} + a_{jj} - a_{ij} - a_{ji}\,| =$$

$$\frac{1}{2}\sum_{i,i=1}^{n} |\,p_{ij}\,|(a_{ii} + a_{jj} - a_{ij} - a_{ji}) = C_n(a,\ |\,p\,|)$$

and the proof is finished.

Remark. If $a_{ij} = a_i b_j$, $p_{ij} = p_i p_j$ the inequality (14) becomes (2).

It is known that if a_{ij} satisfies relation (13) then (see [5], Corolary 3):

(15) $$C_n(a,\ p) \geq C_{n-1}(a,\ p) \geq \ldots \geq C_2(a,\ p) \geq 0$$

where p_{ij} are symmetric and nonnegative.

The following result is also valid.

THEOREM 5. *Let a_{ij}, p_{ij} $(1 \leq i,\ j \leq n)$ be as in Theorem 4. Then the following inequality holds:*

(16) $$C_n(a,\ |\,p\,|) - C_{n-1}(a,\ |\,p\,|) \geq |\,C_n(a,\ p) - C_{n-1}(a,p)\,| \geq 0.$$

The proof follows to the next identity (see [5]):

$$C_n(a,\ p) - C_{n-1}(a,\ p) = \sum_{i=1}^{n-1} p_{in}(a_{ii} - a_{ni} - a_{in} + a_{nn})$$

and we omit the details.

Now, we shall give the integral variant of inequality (14).

Let consider the expression:

$$C(F,\ p) := \int_a^b\!\!\int_a^b p(x,\ y)F(y,\ y)\,dx\,dy - \int_a^b\!\!\int_a^b p(x,\ y)F(x,\ y)\,dx\,dy$$

where p and F are integrable functions on $I^2 = [a,\ b] \times [a,\ b]$. For some generalizations of Ostrowski theorem [3] in connection with this mapping see [5] where further references are given.

A function of two variables $F(x,\ y)$ is said to be a positive set function if:

(17) $$F(x + h,\ y + k) - F(x + h, y) - F(x, y + k) + F(x, y) \geq 0$$

for $h,\ k \geq 0$ (or $h,\ k \leq 0$) with $x + h,\ y + k \in [a,\ b]$ and for arbitrary choices of $x,\ y(x,\ y \in [a,\ b])$. Note that when F has continuous second partial derivatives, condition (17) is equivalent to $\partial^2 F(x,\ y)/\partial x \partial y \geq 0$ (see for example [5]).

THEOREM 6. *Let F be a positive set function and p be symmetric integrable function on $[a,\ b] \times [a,\ b]$. Then the following inequality holds:*

(18) $$C(F,\ |p\,|) \geq |C(F,\ p)\,| \geq 0$$

Proof. We start to the following integral identity:

$$C(F,\ p) = \frac{1}{2}\int_a^b\!\!\int_a^b p(x,\ y)(F(x,\ x) + F(y,\ y) - F(x,\ y) - F(y,\ x))\,dx\,dy$$

Since F is a positive set function, then

$F(x, x) - F(x, y) - F(y, x) + F(y, y) \geqslant 0$ for all $x, y \in [a, b]$ which implies :

$$| C(F,p) | \leqslant \frac{1}{2} \int_a^b\int_a^b | p(x, y) | | F(x, x) + F(y, y) - F(x, y) - F(y, x) | \mathrm{d}x\mathrm{d}y = \frac{1}{2} \int_a^b\int_a^b | p(x, y) | (F(x, x) + F(y, y) - F(x, y) - F(y, x))\mathrm{d}x\mathrm{d}y = C(F, | p |)$$

and the proof is finished.

Remark. If we put in the above theorem $p(x, y) = p(x)p(y)$, $F(x, y) = f(x)g(y)$, $x, y \in [a, b]$, we obtain Theorem 2.

Now, we give the following definition. The n-tuples of real numbers $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ will be called separated if there exists a constant k such that : $a_i \geqslant (\leqslant) k \geqslant (\leqslant) b_j$ for all $i, j = 1, \ldots, n$.

Let also consider the expression :

$$K_n(a, b, p) := \left(\sum_{i=1}^n p_i a_i\right)^2 + \left(\sum_{i=1}^n p_i b_i\right)^2 - \sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i$$

where $p = (p_1, \ldots, p_n)$. The following theorem holds :

THEOREM 7. *Let a, b be two separated n-tuples of real numbers and p be an n-tuple of real numbers. Then*

$$K_n(a, b, |p|) \geqslant | K_n(a, b, p) | \geqslant 0. \tag{19}$$

Proof. It is easy to see that for all $q = (q_1, \ldots, q_n)$ we have

$$K_n(a, b, q) = \frac{1}{2} \sum_{i,j=1}^n q_i q_j (a_i - b_j)(a_j - b_i).$$

Since a, b are separated, we have :

$$(a_i - b_j)(a_j - b_i) \geqslant 0 \text{ for all } i, j = 1, \ldots, n,$$

hence

$$| K_n(a, b, p) | \leqslant \frac{1}{2} \sum_{i,j=1}^n | p_i p_j | | (a_i - b_j)(a_j - b_i) | = \frac{1}{2} \sum_{i,j=1}^n | p_i p_j | (a_i - b_j)(a_j - b_i) = K_n(a, b, | p |)$$

and the theorem is proven.

A similar result to that in Theorem 7 can be obtained for integrals, but we omit the details.

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