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SOME INEQUALITIES FOR α-CONVEX FUNCTIONS

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1. Let $f: R^k \to R$, $x_i \in R^k (i = 1, \ldots, n)$, $x = (x_1, \ldots, x_n)$, $f(x) = (f(x_1), \ldots, f(x_n))$, $S_n(f(x)) = \sum_{i>j} (f(x_i) - f(x_j))^2$. We denote $\langle x_i, x_j \rangle$ the scalar product, $\|x_i\| = \langle x_i, x_i \rangle^{1/2}$ the Euclidean norm, $\nabla f(x_i)$ the gradient, and $\triangle^2 f(x_i)$ the hessian of f in x_i.
A function belongs to the class $C^{1,1}$ if its gradient satisfies Lipschitz condition (with a constant L).

Some inequalities from Andrica-Rașa [1], and Rașa [3], for example

$$\frac{a}{2n^2} S_n(\ln x) \leqslant A_n - G_n \leqslant \frac{b}{2n^2} S_n(\ln x), \quad x_i \in [a, b] \subset R_+^1 \tag{1}$$

we consider as inequalities for α-convex functions. Similarly, some well-known inequalities for convex functions can be refined, and some can be analogously given for nonconvex function.

2. Call $f(x)$ α-convex on a convex set $C \subseteq R^k$ if there exists $\alpha \in R$ such that for every $x, y \in C$, $\lambda \in [0, 1]$ holds

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda) f(y) + \alpha\lambda(1 - \lambda)\|x - y\|^2 \tag{2}$$

If $\alpha > 0$ then $f(x)$ is weakly convex, if $\alpha < 0$ then $f(x)$ is strongly convex and for $\alpha = 0$ the function is convex. Every strongly convex is convex, and every convex is weakly convex.

The following Theorems will be used :

THEOREM 1. [4] *$f(x)$ is α-convex if and only if the function $f(x) + \alpha\|x\|^2$ is convex.*

THEOREM 2. [6] *A differentiable function $f(x)$ is α-convex if and only if for every $x, y \in C$ holds*

$$f(y) - f(x) \geqslant \langle \nabla f(y), y - x \rangle - \alpha\|x - y\|^2$$

THEOREM 3. [6] *Let $f(x)$ be twice continuously differentiable function on a convex nonempty interior set C. Then $f(x)$ is α-convex if and only if for any $x \in C$, and $y \in R^k$*

$$\langle \nabla^2 f(x) y, y \rangle \geqslant -2\alpha\|y\|^2.$$

THEOREM 4. *If $f(x)$ belongs to the class $C^{1,1}$ on a convex set C, then functions $f(x)$ and $-f(x)$ are α-convex with $\alpha = L/2$.*

Proof. This result we get from the inequality ([5, p. 93])

$$|f(x) - f(y) - \langle \nabla f(y),\ x - y\rangle| \leqslant (L/2)\|x - y\|^2$$ and Theorem 2.

3. From Theorem 1 there immediately follows the Jensen inequality for α-convex functions.

THEOREM 5. *Let $f: C \to R$ be α-convex on a convex set $C \subseteq R^k$, and $x_i \in C$ $(i = 1, \ldots, n)$. Then*

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leqslant \frac{f(x_1) + \ldots + f(x_n)}{n} + \frac{\alpha}{n^2} S_n(\|x\|). \tag{3}$$

COROLLARY 1. *Let $f(x)$ be twice continuously differentiable on a convex nonempty interior set C, and let for any $x \in C$, $y \in R^k$ be $-m\|y\|^2 \leqslant \langle \nabla^2 f(x)y, y\rangle \leqslant M\|y\|^2$. Then for every $x_i \in C$ holds*

$$\frac{f(x_1) + \ldots + f(x_n)}{n} - \frac{M}{2n^2} S(\|x\|) \leqslant f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leqslant$$
$$\leqslant \frac{f(x_1) + \ldots + f(x_n)}{n} + \frac{m}{2n^2} S_n(\|x\|). \tag{4}$$

Proof. $f(x)$ is $(m/2)$-convex, and $-f(x)$ is $(M/2)$-convex, by Theorem 3.

Example 1. For $f(t) = \ln t$, $t \in [a, b] \subset R^1_+$ hold $m = 1/a^2$ and $M = -1/b^2$, and for $f(t) = e^t$, $m = -e^a$, $M = e^b$, such that we get the inequalities (1), (2) by Rașa [3].

Example 2. Let $x_i \in [a, b] \subset R^1_+$ and $r \in R$, then for $0 \leqslant r \leqslant 1$ or $r \geqslant 2$ holds

$$\frac{r(r-1)}{2n^2} a^{r-2} S_n(x) \leqslant \frac{x_1^r + \ldots + x_n^r}{n} - \left(\frac{x_1 + \ldots + x_n}{n}\right)^r \leqslant \frac{r(r-1)}{2n^2} b^{r-2} S_n(x) \tag{5}$$

The inequality (5) holds for $1 < r < 2$ or $r < 0$ if a and b change their places.

Remark. $f(t) = t^r$ for $r > 2$ is α-convex with $\alpha = \frac{r(r-1)}{2} a^{r-2}$, etc.

Example 3. $f(t) = \sin t$, $t \in R$ fulfils the conditions of Corollary 1. with $m = M = 1$, and with $m = 1$, $M = 0$ on $[0, \pi]$. Hence

$$\frac{\sin x_1 + \ldots + \sin x_n}{n} - \frac{1}{2n^2} S_n(|x|) \leqslant \sin \frac{x_1 + \ldots + x_n}{n} \leqslant$$
$$\leqslant \frac{\sin x_1 + \ldots + \sin x_n}{n} + \frac{1}{2n^2} S_n(|x|),$$ anf for $x_i \in [0, \pi]$

$$\frac{\sin x_1 + \ldots + \sin x_n}{n} \leqslant \sin \frac{x_1 + \ldots + x_n}{n} \leqslant \frac{\sin x_1 + \ldots + \sin x_n}{n} + \frac{1}{2n^2} S_n(x).$$

Example 4. Let x_i $(i = 1, \ldots, n)$ be positive numbers and $x_1 + \ldots + x_n = nx_0$, then ([2, p. 285])
$\Gamma(x_1) \ldots \Gamma(x_n) \geqslant \Gamma^n(x_0)$, where Γ is the gamma function. However, $f(t) = \ln \Gamma(t)$ is α-convex on $[a, b] \subset R^1_+$ with $\alpha = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(b+k)^2}$, and $f(t) = -\ln \Gamma(t)$ with $\alpha = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(a+k)^2}$, such that (Theorem 5) for $x_i \in [a, b] \subset R_+$ and $x_1 + \ldots + x_n = nx_0$ holds

$$e^{\frac{S_n(x)}{2n} \sum_{k=0}^{\infty} \frac{1}{(b+k)^2}} \cdot \Gamma^n(x_0) \leqslant \Gamma(x_1) \ldots \Gamma(x_n) \leqslant e^{\frac{S_n(x)}{2n} \cdot \sum_{k=0}^{\infty} \frac{1}{(a+k)^2}} \Gamma^n(x_0).$$

4. Let $f(x)$ be α-convex on $[a, b]$. Then the Hadamard inequality holds

$$f\left(\frac{a+b}{2}\right) + \frac{\alpha}{12}(b-a)^2 \leqslant \frac{1}{b-a} \int_a^b f(x)\,dx \leqslant \frac{f(a) + f(b)}{2} + \frac{\alpha}{6}(b-a)^2.$$

Example 5. $f(t) = 1/t$ is α-convex on $[1, x]$ with $\alpha = -1/x^3$, and $f(t) = -1/t$ is 1-convex. For $x \geqslant 1$ hold

$$\frac{x^2 - 1}{2x} - \frac{(x-1)^3}{6} \leqslant \ln x \leqslant 2\frac{x-1}{x+1} + \frac{(x-1)^3}{12}, \tag{6}$$

$$2\frac{x-1}{x+1} + \frac{(x-1)^2}{12x^3} \leqslant \ln x \leqslant \frac{x^2 - 1}{2x} - \frac{(x-1)^3}{6x^3}. \tag{7}$$

Replacing x with $1 + \frac{1}{x}$ and combining (6) and (7) we get for $x > 0$

$$\frac{2}{2x+1} + \frac{1}{12(x+1)^3} \leqslant \ln\left(1 + \frac{1}{x}\right) \leqslant \frac{2}{2x+1} + \frac{1}{12x^3} \tag{8}$$

The right part of the inequality ([2, p. 274]) $\frac{2}{2x+1} \leqslant \ln\left(1 + \frac{1}{x}\right) \leqslant$ $\leqslant \frac{2}{2x+1} + \frac{1}{6x(x+1)(2x+1)}$ is less than the right part of (8). But their difference tends to zero when x tends to infinity.

Example 6. For $x \in [0, \pi/2)$ holds

$$\cos\frac{x}{2} - \frac{x^2}{24} \leqslant \frac{\sin x}{x} \leqslant \cos\frac{x}{2} - \frac{x^2}{24}\cos x.$$

In [2, p. 236] the following inequalities were given for $x \in [0, \pi/2)$.

$$\cos x \leqslant \frac{\cos x}{1 - \frac{x^2}{3}} \leqslant \sqrt[3]{\cos x} \leqslant \cos\frac{x}{\sqrt{3}} \leqslant \frac{\sin x}{x} \leqslant \cos\frac{x}{2} \leqslant 1.$$

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