MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION,
Tome 19, N° 1, 1990, pp. 85—91

ON A QUASIMONOTONIC MAX-MIN PROBLEM

STEFAN ŢIGAN
(Cluj-Napoca)

1. Introduction. The paper is concerned to a max-min problem having a quasimonotonic objective function and linear constraints. This problem represents a generalizations of some max-min problems considered in the literature in which the objective function was assumed to be linear [7], linear-fractional [6] or polynomial linear-fractional [10]. It is shown that a quasimonotonic max-min problem can be reduced to a quasiconvex programming problem having at least an optimal solution which is a vertex of the feasible set. A finite algorithm for solving this problem is suggested.

2. Problem formulation. In this paper we consider the following max-min problem:

(QM) Find

$$V = \max \min f(x, y) \tag{2.1}$$

subject to:

$$Ax + By \leqslant b, \tag{2.2}$$

$$Cx \leqslant c, \quad x \geqslant 0, \tag{2.3}$$

$$Dy \leqslant d, \quad y \geqslant 0, \tag{2.4}$$

where $x \in R^p$, $y \in R^q$ are decision variable vectors while A, B, C, D are $m \times p, m \times q, m_1 \times p, m_2 \times q$ given matrices respectively, $b \in R^m$, $c \in R^{m_1}$, $d \in R^{m_2}$ are given right hand side vectors and f is a quasimonotonic function on the feasible set $S \subset R^p \times R^q$,

$$S = \{(x, y) \in R^p \times R^q : Ax + By \leqslant b, \quad Cx \leqslant c, \; Dy \leqslant d, \; x \geqslant 0, \; y \geqslant 0\} \tag{2.5}$$

We recall that the function $h : W \to R$ is *quasiconvex* on the convex set $W \subset R^n$ if for all $w', w'' \in W$ and for all $t \in (0, 1)$:

$$h(tw' + (1 - t)w') \leqslant \max \{h(w'), h(w'')\}. \tag{2.6}$$

The function h is *strictly quasiconvex* on W if for all w', $w'' \in W$, such that $h(w') \neq h(w'')$, and for all $t \in (0,1)$ the inequality (2.6) is strictly satisfied.

The function h is quasiconcave on W if $-h$ is quasiconvex on W.

The function h is *quasimonotonic* on W if it is both quasiconvex and quasiconcave on W.

For further discussion regarding optimality conditions and techniques for solving usual quasimonotonic programming problems with linear constraints, see references [2], [3], [5], [8], [9], [11], [12], [14].

The problem (QM) is an extension of the linear max-min problem considered by Falk [7] or of the linear-fractional max-min problem studied by Cook, Kirby and Mehndiratta [6]. The polynomial linear-fractional max-min problem considered by Stancu-Minasian and Patkar [10] is a particular quasimonotonic max-min problem too. From the problem (QM) it can be also obtained by suitable particularization the quasimonotonic max-min problem with joint constraints (Ţigan [13]) or with independent constraint sets for the variable vectors x and y (Belenkiy [4]).

We introduce now as in [13] (see, also [6] and [7]) some notation and terminology that will be used in the remainder of the paper.

We denote the projections of S on R^p and R^q by:

(2.7) $$X = \{x \in R^p : \exists y \geqslant 0 \text{ such that } Ax + By \leqslant b,\ Cx \leqslant c,\ Dy \leqslant d,\ x \geqslant 0\}$$

and

(2.8) $$Y = \{y \in R^q : \exists x \geqslant 0 \text{ such that } Ax + By \leqslant b,\ Cx \leqslant c,\ Dy \leqslant d,\ y \geqslant 0\},$$

respectively.

DEFINITION 1. A point y' is optimal with respect to x' if $(x', y') \in S$ and y' is an optimal solution of the problem:

(2.9) $$\min\{f(x', y) : y \in T(x')\}.$$

where

(2.10) $$T(x') = \{y \in R^q : By \leqslant b - Ax', Dy \leqslant d, y \geqslant 0\}.$$

DEFINITION 2. A point (x', y') is an optimal solution of (QM) if:

(i) y' is optimal with respect to x';

(ii) $f(x', y') \geqslant \min\{f(x, y) : y \in T(x)\}, \quad \forall x \in X.$

3. Main results. In this section some properties for the problem (QM), which generalize similar results from [6], [7] and [13] are developed.

We shall assume throughout the remainder of the paper that:

H1) S is a nonempty and compact set;

H2) f is a quasimonotonic function on the polyhedral convex set S.

Under the assumptions H1 and H2, we can define a function $F : X \to \mathbb{R}$, by

(3.1) $$F(x) = \min\{f(x, y) : y \in T(x)\}, \qquad \forall x \in X,$$

where $T(x)$ is given by (2.10).

The first theorem is useful to characterize problem QM as a problem involving the maximization of a quasiconvex function over a linear polyhedron.

THEOREM 1. ([13]) *If the assumptions H1 and H2 hold, then the function F is quasiconvex on X.*

Proof. In order to show that F is a quasiconvex function, for some points x^1, x^2 in X and $t \in [0,1]$, let y^i be any optimal solution of the problem:

(3.2) $$F(x^i) = \min\{f(x^i, y) : y \in T(x^i)\}, \quad i = 1, 2.$$

Therefore, we have

(3.3) $$F(x^i) = f(x^i, y^i), \quad i = 1, 2,$$

and

(3.4) $$F(tx^1 + (1-t)x^2) = \min\{f(tx^1 + (1-t)x^2, y) : y \in T(x(t))\},$$

where $x(t) = tx^1 + (1-t)x^2$.

Since $y(t) = ty^1 + (1-t)y^2$ is a feasible solution for the problem (3.4), i.e. $y(t) \in T(x(t))$, it follows that:

(3.5) $$F(tx^1 + (1-t)x^2) \leqslant f(tx^1 + (1-t)x^2, ty^1 + (1-t)y^2).$$

But from the quasiconvexity of f (see (2.6)) and by the definition of the points $y^i (i = 1, 2)$ (see, (3.3)), we have:

(3.6) $$f(tx^1 + (1-t)x^2, ty^1 + (1-t)y^2) \leqslant \max\{f(x^1, y^1), f(x^2, y^2)\} = \max\{F(x^1), F(x^2)\}.$$

Then combining (3.5) and (3.6), we get that F is quasiconvex.

THEOREM 2. *If H1 holds and f is a quasimonotonic and strictly quasiconvex function on S, then F is strictly quasiconvex on S.*

Proof. The proof is similar to that of Theorem 1.

THEOREM 3. *If the assumption H1, H2 hold then there exists an optimal solution (x'', y'') of (QM) such that x'' is a vertex of the polyhedral set X.*

Proof. The theorem follows immediately from the quasiconvexity of the function F (see Theorem 1) and the fact that a quasiconvex function attains its maximum at least at a vertex of a convex polyhedron.

THEOREM 4. *If the assumptions H1, H2 hold then there exists an optimal solution (x'', y'') of (QM) that is a vertex of the polyhedral set S.*

Proof. Let (x'', y'') be an optimal solution for (QM), for which, by Theorem 3, x'' is a vertex of X.

Since any point in S may be written as a convex combination of the vertices of S, let:

(3.7) $$(x'', y'') = \sum_{i=1}^{s} t_i(x^i, y^i),$$

where $(x^1, y^1), \ldots, (x^s, y^s)$ are distinct vertices of S and

$$t_1 + t_2 + \ldots + t_s = 1, \quad t_i \geqslant 0, \quad i \in I = \{1, 2, \ldots, s\}.$$

From (3.7), we have:

$$x'' = \sum_{i=1}^{s} t_i x^i \text{ and } y'' = \sum_{i=1}^{s} t_i y^i.$$

Since x'' is a vertex of X and each $x^i \in X$, it follows that $x^i = x''$, for all $i \in I$. Thus $y^i \in T(x'')$, for all $i \in I$. But because y'' minimizes $f(x'', y)$ over $T(x'')$, we have:

$$(3.8) \qquad f(x'', y'') \leqslant f(x'', y^i), \text{ for all } i \in I.$$

On the other hand, since f is a quasimonotonic function on S, we get:

$$(3.9) \quad f(x'', y'') = f\left(x'', \sum_{i=1}^{s} t_i y^i\right) \geqslant \min\{f(x'', y^i) : i \in I\} = f(x'', y^{i'}),$$

for some $i' \in I$.

Therefore, from (3.8) and (3.9) it follows that $f(x'', y'') = f(x'', y^{i'})$, i.e., the vertex $(x', y') = (x'', y^{i'})$ of S is an optimal solution for the problem (QM).

Now, as in the linear case [7], the property that max-min operator may yield a higher value of the optimal value than does min-max operator is also true for quasimonotonic case.

THEOREM 5. *If H1 and H2 hold, then*:

$$v^\circ = \max_x \min_y \{f(x, y) : (x, y) \in S\} \geqslant v_0 = \min_y \max_x \{f(x, y) : (x, y) \in S\}.$$

Proof. For any fixed $x \geqslant 0$, the set $T(x)$ is a subset of Y. Hence, we have:

$$(3.10) \quad \min\{f(x, y) : y \in T(x)\} \geqslant \min\{f(x, y) : y \in Y\}, \ \forall x \in X,$$

so that:

$$v^\circ = \max_{x \in X} \min_{y \in T(x)} f(x, y) \geqslant \max_{x \in X} \min_{y \in Y} f(x, y).$$

Then, by using min-max theorem, we have:

$$(3.11) \quad \max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y) \geqslant$$

$$\geqslant \min_{y \in Y} \max_{x \in L(y)} f(x, y) = v_0,$$

where the last inequality follows from the fact that:

$$L(y) = \{x \in R^p : Ax \leqslant b - By, \quad Cx \leqslant c, \quad x \geqslant 0\} \subset X,$$

for any given $y \in Y$.

But from (3.10) and (3.11) it results that $v^\circ \geqslant v_0$.

Now, let us consider the particular case of the problem (QM) with separate constraints, that is, the case when only the constraints (2.3) and (2.4) are present. In this case, we have:

$$X = \{x \in R^p : Cx \leqslant c, \quad x \geqslant 0\},$$

$$Y = \{y \in R^q : Dy \leqslant d, \quad y \geqslant 0\},$$

and $T(x) = Y$, for all x in X.

Then we obtain the following version of Theorem 1.

THEOREM 6. *If the problem (QM) has separate constraints and the assumptions H1 and H2 hold, then the function F is quasi monotonic on X.*

Proof. Let Y_e be the set of all extremal points of Y. The function F is quasiconcave on X, since:

$$F(x) = \min\{f(x, y) : y \in Y\} = \min\{f(x, y) : y \in Y_e\}, \quad \forall x \in X,$$

that is the minimum of finite family of quasiconcave functions

$$\{g_y : y \in Y_e\}, \text{ where } g_y : S \to R \text{ and } g_y(x) = f(x, y).$$

On the other hand, by Theorem 1, F is quasiconvex on X. Hence F is quasimonotonic on X.

From Theorem 1, it results that the problem (QM) can be reduced to a quasiconvex programming problem, while, by Theorem 6, the problem (QM) with separate constraints is equivalent with a quasimonotonic programming problem with linear constraints.

Unfortunately, we can not apply the linearization algorithm (see [2], [12]) to solve this problem, because the objective function F is not generally differentiable over the set X.

4. Algorithm development. Theorem 4 shows that an optimal solution of (QM) occurs at an extreme point of the feasible set S. Since the set S^e of all extreme points of the polyhedral set S is finite, the problem (QM) could be solved by a finite algorithm (see [1]) that enumerates and compares all basic feasible solutions in the convex polyhedron S.

However, from Theorems 1 and 4, it follows that the branch-and-bound procedure described in [6] (see, also [10]) can be extended for the problem (QM). This procedure is based on the concept of a k-th best extreme point solution and its broad outline is given below.

Since S^e contains a finite number of points, the function f assumes a finite number of values over the set S^e. Denote these values by V_k, $k = 1, 2, \ldots, r$, where $V_k \geqslant V_{k+1}$, $k = 1, 2, \ldots, r - 1$. The subset of S^e containing all elements (x, y) for which $f(x, y) = V_k$, will be referred as the k-th best extreme point solutions of the quasimonotonic problem:

$$(4.1) \qquad \max\{f(x, y) : (x, y) \in S^e\}.$$

Note that for the quasimonotonic programming problems there is at least an optimal solution, which is a vertex of the feasible region (see,

[8]). Based on this fact, for these problems, some simplex linearization techniques was considered (see, [2], [5], [12]).

The k-th iteration of the algorithm will consist of the following steps:

1. Find the set T_k^e of k-th best extreme point solutions of the problem (4.1).

2. (i) If there is an element (x^k, y^k) in T_k^e such that y^k is optimal with respect to x^k(see, definition 1), then (x^k, y^k) is an optimal solution of (QM) and stop.

(ii) Otherwise, go to the next iteration, by returning to the step 1, where the set T_{k+1}^e is generated.

In order to carry out the algorithm the following nonlinear quasimonotonic programming problems must be solved:

$$\max\{f(x, y) : A_1x + B_1y + z = b_1,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}, \tag{4.2}$$

$$\max\{f(x, y) : A_1x + B_1y + z = b,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0,\ x_{j_t},\ t = 2, \ldots, q \text{ nonbasic}\}, \tag{4.3}$$

where z is a vector of slacks and x_{j_t} are the components of the $(p + q + m + m_1 + m_2)$-vector (x, y, z) and

$$A_1 = \begin{pmatrix} A \\ C \end{pmatrix},\quad B_1 = \begin{pmatrix} B \\ D \end{pmatrix},\quad b_1 = \begin{pmatrix} b \\ c \\ d \end{pmatrix}.$$

Moreover, the remarks concerning the finiteness of the algorithm given in [6] for the fractional max-min problem, also hold for the quasimonotonic problem.

REFERENCES

1. Balinski M. L., *An algorithm for finding all vertices of convex polyhedral sets*, J. Soc. Indust. Appl. Math., 9(1961), 72–88.
2. Bector C. R., Jolly P. L., *Programming problems with pseudo-monotonic objectives*, Math. Operationsforsch., und Statists., ser. Optimization, 15(1984), 2, 217–229.
3. Belenkiy A. S., *Minimization of a monotone function on a polyhedral set*, Avt. i Telemeh., 9(1982), 112–121.
4. Belenkiy A. S., *On minimax problems with monotone functions on polyhedral sets*, Avt. i Telemeh., 10(1982), 91–106.
5. Bhatt S. K., *Linearization technique for linear-fractional and pseudomonotonic programs revisited*, Cahiers du C.E.R.O., 23(1981), 1, 53–56.
6. Cook W. D., Kirby M. J. L., Mehndiratta S. L., *A linear-fractional Max-min problem*, Operations Research, 23(1975), 3, 511–521.
7. Falk J. E., *A linear max-min problem*, Math. Progr. 5(1973), 169–188.
8. Martos B., *Nonlinear Programming*, North-Holland Pub. Co., 1975.
9. Mond B., *Techniques for pseudomonotonic programming*, La Trobe University, Melbourne, Pure Math. Res. Paper, No. 82–12, 1982.
10. Stancu-Minasian I. M., Patkar V., *A note on nonlinear fractional max-min problem*, Nat. Acad. Sci. Letters (India), 8(1985), 2, 39–41.
11. Țigan S., *Sur une méthode de décomposition pour le problème de programmation monotone*, Mathematica, 13(36) (1971), 2, 347–354.

12. Țigan S., *On the linearization technique for quasimonotonic optimization problems*, Rev. Analyse Numer. Théor. Approx., 12(1983), 1, 89–96.
13. Țigan S. *A quasimonotonic max-min programming problem with linked constraints*, Itinerant Sem. on functional eq., approx. and conv., Cluj-Napoca, University, (1986), 279–284.
14. Weber R., *Pseudomonotonic multiobjective Programming*, Discussion Paper, Institute of Operations Research, University of Saarland, Saarbruecken, B 8203, 1982.

Received 19.X.1989

Centrul teritorial de calcul electronic
3400 Cluj-Napoca
România