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ON A THEOREM OF V. N. NIKOLSKI ON CHARACTERIZATION OF BEST APPROXIMATION FOR CONVEX SETS

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1. Introduction. The aim of this paper is to study the relations between two theorems of characterization of best approximation for convex sets proved by V. N. Nikolski [13], [14] and S. A. Azizov [2] and to extend Nikolski's theorem to the case of best approximation by elements of p-convex sets.

2. Characterizations of elements of best approximation by elements of convex sets. Let X be a normed space (over R or C) and let X^* be its conjugate space with elements denoted by $f, g, \ldots$. Let $S^* = \{f \in X^* : \|f\| = 1\}$ denote the unit sphere of X and B^* the unit ball of X^*.

A subset Γ of S^* is called *fundamental* if

(2.1) $\qquad \Gamma$ is w^*-closed,

and

(2.2) $\forall\, x \in X \quad \exists f_x \in \Gamma$ such that $\|x\| = |f_x(x)| = \sup\{|f(x)| : f \in \Gamma\}$ (the symbol w^* will refer always to the $\sigma(X^*, X)$-topology of the space X^*).

Obviously, the whole unit sphere S^* is a fundamental set but, as we shall see by an example, there are proper subsets of S^* which are fundamental. A possible candidate for Γ would be the set extB^* of the extreme points of B^*. In this case the condition (2.2) is verified but the set extB^* is not always w^*-closed. For this reason, there are two type characterization theorems of the best approximation elements for convex sets :

— Theorems giving characterizations in terms of some fundamental subsets of the unit sphere S^*, on the line of V.N. Nikolski [13], [14], and

— Theorems giving characterizations in terms of the extreme points of the unit ball B^* of X^*, as given by A. L. Garkavi [7] and by I. Singer [17] (see also [18], Chapter I, Th. I.13 and Appendix I, §1).

In this paper we shall be concerned with theorems of the first type. Those involving extremal points of the unit ball of X^* will be considered in a subsequent paper.

As usually, for a nonvoid subset Y of a normed space X and $x \in X$ denote by $d(x, Y) = \inf\{\|x - y\| : x \in Y\}$ the *distance* from x to Y, and by $P(x) = \{y \in Y : \|x - y\| = d(x, Y)\}$ — the (possible void) set of *nearest points* to x in Y (the elements of $P_Y(x)$ are called also *projections* of x onto Y, or *elements of best approximation* of x by elements in Y).

Now, we can state the characterization theorems:

2.1. THEOREM (V. N. Nikolski [13], [14]). *Let Y be a nonvoid subset of a normed space X, Γ a fundamental subset of S^*, $x \in X \setminus Y$ and $y_0 \in Y$. In order that y_0 be a nearest point to x in Y it is sufficient and, if Y is convex, also necessary that for every $y \in Y$ to exist a functional $f = f_y \in \Gamma$, such that*

$$\text{(N1)} \qquad |f(x - y_0)| = \|x - y_0\|$$

and

$$\text{(N2)} \qquad \operatorname{Re}\,[f(y_0 - y)\overline{f(x - y_0)}] \geq 0.$$

2.2. THEOREM. (S. A. Azizov [2]). *Let Y be a nonvoid convex subset of a normed space X, Γ a fundamental subset of S^* and $x \in X \setminus Y$. An element $y_0 \in Y$ is a nearest point to x in Y if and only if for every $y \in Y$ there exists a functional $f = f_y$ such that*

$$\text{(A1)} \qquad |f(x - y_0)| = \|x - y_0\|$$

and

$$\text{(A2)} \qquad \operatorname{Re}[f(x - y)\overline{f(x - y_0)}] > 0$$

The following proposition shows that the necessity part of Azizov's theorem is an immediate consequence of Nikolski's theorem. The Example 2.6.b) shows that conditions (A1), (A2), are not always sufficient in order that y_0 be a nearest point to x in Y.

2.3. PROPOSITION. *Let X be a normed space, $y_0, y \in X$ and $x \in X \setminus \{y_0, y\}$. If the functional $f \in S^*$ verifies conditions (N1), (N2), then f verifies also conditions (A1), (A2).*

Proof. Suppose $f \in S^*$ verifies conditions (N1), (N2). As condition (A1) is the same as (N1), it remains to prove that f verifies also condition (A2), which follows from the following calculations:

$$\operatorname{Re}[f(x - y)\overline{f(x - y_0)}] = \operatorname{Re}[(f(x - y_0) + f(y_0 - y))\overline{f(x - y_0)}] =$$

$$= |f(x - y_0)|^2 + \operatorname{Re}[f(y_0 - y)\overline{f(x - x_0)}] \overset{(N1)}{=} \|x - y_0\|^2 +$$

$$+ \operatorname{Re}[f(y_0 - y)\overline{f(x - x_0)}] \overset{(N2)}{\geq} \|x - y_0\|^2 > 0.$$

Let's also mention the following result:

2.4 THEOREM. (A. L. Garkavi [6]) G. Sh. Rubinshteĭn [16]), *Let X be a normed space, Y a nonvoid convex subset of X and $x \in X \setminus Y$. An*

element $y_0 \in Y$ is a nearest point to x in Y if and only if there exists a functional $f \in S^*$ such that

$$\text{(G1)} \qquad |f(x - y_0)| = \|x - y_0\|$$

and

$$\text{(G2)} \qquad \operatorname{Re} f(y_0 - y) \geq 0, \quad \forall\, y \in Y.$$

2.5. *Remark.* In this case the characterizing functional f does not depend on the point $y \in Y$ and, geometrically, conditions (G1), (G2) mean that the hyperplane $H = \{z \in X : \operatorname{Re} f(x - z) = \|x - y_0\|\}$ supports the closed ball $B(x, \|x - y_0\|)$ and the set Y at the same point y_0 and separates them. Indeed

$\operatorname{Re} f(x - z) \leq |f(x - z)| \leq \|x - z\| \leq \|x - y_0\|$,

for all $z \in (x, \|x - z_0\|)$, and

$$0 \leq \operatorname{Re} f(y_0 - y) \leq \operatorname{Re} f(y_0 - x) + \operatorname{Re} f(x - y) =$$

$$= -\|x - y_0\| + \operatorname{Re} f(x - y), \text{ implying}$$

$$\operatorname{Re} f(x - y) \geq \|x - y_0\|, \quad \text{for all } y \in Y.$$

This geometrical interpretation of Garkavi's theorem was given by V. N. Burov [3].

The following example shows that the characterizing functional f_y in Theorem 2.1, may effectively depend on the point $y \in Y$ and that conditions (A1), (A2), from Theorem 2.2, do not ensure that y_0 is a nearest point to x in Y.

2.6. *Example* a) We shall consider the case of real scalars, so that the conjugation bar and the symbol Re have no meaning and will be omitted everywhere.

Take $X = R^2$ with the l^1-norm, $\|x\| = |x_1| + |x_2|$, for $x = (x_1, x_2) \in X$, $Y = \{x \in X : x = (x_1, x_2), (x_1 + 2)^2 + x_2^2 \leq 1\}$, $y_0 = (-1, 0)$ and $\bar{x} = (0, 0)$ — the approximated element.

The conjugate space of X is $X^* = R^2$ with the l_∞-norm $\|f\| = \max\{|f_1|, |f_2|\}$, for $f = (f_1, f_2) \in R^2$ and $f(x) = f_1x_1 + f_2x_2$, for $x \in X$.

Take $\Gamma = \{f, g\}$, where $f = (1, 1)$ and $g = (-1, 1)$. As $|f(\bar{x} - y_0)| = \|\bar{x} - y_0\|$ and $|g(\bar{x} - y_0)| = \|\bar{x} - y_0\|$, condition (N1) is automatically verified.

For $y \in Y$ put $\Gamma_y = \{h \in \Gamma : h(y_0 - y)f(\bar{x} - y_0) \geq 0\}$, $H_1 = \{x \in R^2 : f(x) - f(y_0) = 0\} = \{x \in R^2 : x = (x_1, x_2), x_1 + x_2 = -1\}$ and $H_2 = \{x \in R^2 : g(x) - g(y_0) = 0\} = \{x \in R^2 : x = (x_1, x_2), -x_1 + x_2 = -1\}$. Let H_i^-, H_i^+, $i = 1, 2$ denote the corresponding closed semispaces. Since $f(\bar{x} - y_0) = 1$ and $g(\bar{x} - y_0) = -1$, it follows that $f(y_0 - y)f(\bar{x} - y_0) \geq 0 \Leftrightarrow f(y_0 - y) \geq 0 \Leftrightarrow y \in H_1^-$ and

$$g(y_0 - y)g(\bar{x} - y_0) \geq 0 \Leftrightarrow g(y_0 - y) \leq 0 \Leftrightarrow y \in H_2^+.$$

Therefore, we have to consider the following three situations:

(i) if $y \in Y \cap H_1^- \cap H_2^+$ then $\Gamma_y = \{f, g\} = \Gamma$;
(ii) if $y \in Y \cap H_1^+$ then $\Gamma_y = \{g\}$;
(iii) if $y \in Y \cap H_2^+$ then $\Gamma_y = \{f\}$,

which show that the characterizing functional effectively depends on the point $y \in Y$.

b) The hypotheses are the same as in case a. We shall show that there exists a point $y_1 \in Y$, $y_1 \neq y_0$ such that conditions (A1), (A2) from Theorem 2.2 are satisfied. As $y_0 = (-1, 0)$ is the only element of best approximation for $\bar{x} = (0, 0)$ by elements in Y it follows that conditions (A1), (A2), do not characterize the elements of best approximation.

Take $y_1 = (\alpha, \beta)$, where $\alpha = -2 + \sqrt{2}/2$, $\beta = \sqrt{2}/2$. Then $\bar{x} - y_1 = (-\alpha, -\beta)$. The fundamental set $\Gamma = \{f, g\}$ is as above, so that condition (A1) is again automatically verified.

Take a point $y = (\xi, \eta) \in Y$, $\xi = -2 + \rho \cos t$, $\eta = \rho \sin t$, $0 \leqslant \rho \leqslant 1$, $t \in [0, 2\pi]$.

It follows that

$$f(\bar{x} - y)\ f(\bar{x} - y_1) = (2 - \rho \cos t - \rho \sin t)\,(-\alpha - \beta) =$$
$$= (2 - \rho \cos t - \rho \sin t) \cdot 2 > 0$$

and

$$g(\bar{x} - y)\ g(\bar{x} - y_1) = (-2 + \rho \cos t - \rho \sin t)(\alpha - \beta)$$
$$= (-2 + \rho \cos t - \rho \sin t)(-2) > 0,$$

so that condition (A2) is verified by both of f and g for all $y \in Y$.

3. **The case of p-convex sets.** There are many extensions of the notion of convexity. In 1967 J. Ponstein [14] counted seven and, since then, very probably that their number has considerably grown, but it is not our intention to give a detailed account of the present-day situation in convexity theory. In this paper, we shall be concerned with one of this extension, namely with the so-called p-convex sets: For a fixed p, $0 < p \leqslant 1$, a subset Y of a vector space is called *p-convex* if $pY + (1 - p)Y \subset Y$. The notion of p-convex set is a particular case of the more general notion of quasi-convex set considered by J. W. Green and W. Gustin [8]: A subset Y of a vector space is called *quasi-convex* if together with any of its points x, y it contains also all the points dividing the segment $[x, y]$ into a ratio belonging to a prescribed set $\Delta \subset]\ 0,1\ [$ (i.e. $\alpha x + (1 - \alpha)y \in Y$, for all $\alpha \in \Delta$). Obviously, that for $\Delta = \{p\}$ one obtains the notion of p-convex set. Some topological and support properties of p-convex sets were studied in [1] and [11]. Applications to p-convex programming and to duality theory for best approximation by p-convex sets were given in [12] and [4].

The main result about p-convex sets we shall use is the following lemma, which can be easily proved by an induction argument:

3.1. LEMMA. *If Y is a p-convex set and $x, y \in Y$ then $p^n x + (1 - p^n)\, y \in Y$, for all $n \in N$.*

3.2. Proof of Nikolski's theorem for p-convex sets.

Necessity. **Suppose that Y is a p-convex subset of a normed space X, $x \in X \setminus Y$, $y_0 \in Y$ and Γ is a fundamental subset of the unit sphere S^* of the conjugate space X^*. We proceed by contradiction: Let $y_0 \in P_Y(x)$ and suppose that there exists an element $y_1 \in Y$ such that**

(3.1) $$|f(x - y_0)| < \|x - y_0\|$$

or

(3.2) $$\operatorname{Re}\,[f(y_0 - y_1)\,\overline{f(x - y_0)}] < 0,$$

for every $f \in \Gamma$. For $n \in N$ let

(3.3) $$y_{n+1} = p^n y_1 + (1 - p^n) y_0$$

We shall show that there exists $n_0 \in N$, such that

(3.4) $$|f(x - y_{n_0+1})| < \|x - y_0\|$$

for all $f \in \Gamma$, whence, by (2.2) it follows $\|x - y_{n_0+1}\| < \|x - y_0\|$, in contradiction to the hypothesis that y_0 is a nearest point to x in Y.

Consider the following three sets:

$$U = \{f \in \Gamma : \operatorname{Re}\,[f(y_0 - y_1)\,\overline{f(x - y_0)}\,] < 0\}$$

$$\Delta = \Gamma \setminus U = \{f \in \Gamma : \operatorname{Re}[f(y_0 - y_1)\,\overline{f(x - y_0)}] \geqslant 0\}$$

and

$$\Lambda = \{f \in \Gamma : |f(x - y_0)| = \|x - y_0\|\}.$$

By the choice of the element y_1 (conditions (3.1) and (3.2)) we have $\Lambda \subset U$ so that $\Delta \cap \Lambda = \emptyset$. By Alaoglu-Bourbaki theorem, the set B^* is w^*-compact, whence, by (2.1), the set Γ is also w^*-compact. Since Δ is a w^*-closed subset of Γ it follows that Δ has the following properties:

(3.5) the set Δ is w^*-compact,

and

(3.6) $$\forall\, f \in \Delta, \quad |f(x - y_0)| < \|x - y_0\|.$$

By the w^*-continuity of the application $f \to |f(x - y_0)|$, it follows that

(3.7) $$m := \sup\{|f(x - y_0)| : f \in \Delta\} < \|x - y_0\|$$

Take $\varepsilon > 0$ such that

(3.8) $$a := m + \varepsilon < \|x - y_0\|$$

and let V be defined by

(3.9) $$V = \{f \in S^* : |f(x - y_0)| < a\}.$$

It follows that V is a relatively w^*-open subset of S^* and $\Delta \subset V$.

Now, taking into account definition (3.3) of y_{n+1} we obtain

$$|f(x - y_{n+1})| \leqslant p^n |f(x - y_1)| + (1 - p^n)\ |f(x - y_0)| <$$
$$< p^n \|x - y_1\| + (1 - p^n)\ a < \|x - y_0\|,$$

for all $n \geqslant n_1$, where $n_1 \in N$ is chosen such that

$$0 < p^{n_1} < \frac{\|x - y_0\| - a}{\|x - y_1\| - a} < 1. \tag{3.10}$$

It follows that

$$|f(x - y_{n+1})| < \|x - y_0\| \tag{3.11}$$

for all $n \geqslant n_1$ and all $f \in V$.

Now, since V is a relatively w^*-open subset of S^* it follows that the set $W = U \setminus V = \Gamma \setminus V$ has the following properties :

(3.12) the set W is w^*-compact,

and

$$\forall\, f \in W,\ \operatorname{Re}\,[f(y_0 - y_1)\overline{f(x - y_0)}] < 0. \tag{3.13}$$

Therefore

$$b : = \sup\{\operatorname{Re}[f(y_0 - y_1)\overline{f(x - y_0)}] : f \in W\} < 0 \tag{3.14}$$

so that

$$|f(x - y_{n+1})|^2 = [f(x - y_0) + p^n f(y_1 - y_0)] \cdot \tag{3.15}$$
$$\cdot [\overline{f(x - y_0)} + p^n \overline{f(y_1 - y_0)}\ ] = |f(x - y_0)|^2 +$$
$$+ 2p^n \operatorname{Re}\,[f(y_1 - y_0)\overline{f(x - y_0)}\ ] + p^{2n}\,|f(y_1 - y_0)|^2 \leqslant$$
$$\leqslant \|x - y_0\|^2 + 2bp^n + p^{2n}\,\|y_0 - y_1\|^2 < \|x - y_0\|^2,$$

for all $f \in W$ and all $n \geqslant n_2$, where $n_2 \in N$ is chosen such that

$$p^{n_2} < \frac{-2b}{\|y_0 - y_1\|^2}. \tag{3.16}$$

(Obviously that condition (3.2) implies $y_0 \neq y_1$).

Now, combining (3.11) and (3.15), it follows that (3.4) holds for $n_0 = \max\{n_1, n_2\}$, which ends the proof of the necessity part of Theorem 2.1.

Sufficiency. Suppose that Y is an arbitrary subset of a normed space X, $x \in X \setminus Y$ and $y_0 \in Y$. Let y be an element of Y and let $f = f_y \in \Gamma$ verifying conditions (N1), (N2). Then

$$\|x - y_0\|^2 \overset{(N1)}{=} |f(x - y_0)|^2 = f(x - y_0)\ \overline{f(x - y_0)} =$$
$$= \operatorname{Re}[f(x - y)\ \overline{f(x - y_0)}\ ] + \operatorname{Re}\,[f(y - y_0)\ \overline{f(x - y_0)}\ ] \leqslant$$

$$\overset{(N2)}{\leqslant} \operatorname{Re}\,[f(x - y)\overline{f(x - y_0)}\ ] \leqslant |f(x - y)|\,|f(x - y_0)| \leqslant$$
$$\leqslant \|x - y\|\cdot\|x - y_0\|,$$

implying, $\|x - y_0\| \leqslant \|x - y\|$. As the element y was arbitrarily chosen it follows $\|x - y_0\| \leqslant \|x - y\|$, for all $y \in Y$, i.e. $y_0 \in P_Y(x)$. Theorem 2.1 is completely proved.

REFERENCES

1. Aleman, A., *On some generalizations of convex sets and convex functions*, Anal. Nume , Théor. Approx. **14** (1985), 1–6.
2. Azizov, A. S., *New criteria for the element of best approximation for convex sets* (in Russian), Doklady Akad. Nauk SSR **27**(1984), 615–618.
3. Burov, V. N., *Approximation with restrictions in normed linear spaces* (in Russian), Ukrain. Mat. J. **15**(1963), 135–144.
4. Cobzaş, S. and Muntean, I., *Duality relations and characterizations of best approximation for p-convex sets*, Anal. Numer. Théor. Approx. **16**(1987), 95–108.
5. Dunford, M. and Schwartz, J. T., *Linear Operators*. I Interscience Publ., New York, 1958.
6. Garkavi, A. L., *Duality theorems for approximation by elements of convex sets* (in Russian), Uspekhi Mat. Nauk **16**(1961), 141–145.
7. Garkavi, A. L., *On a criterion for best approximation*, (in Russian) Siberian Mat., J., **5**(1964), 472–476.
8. Green, J. W. and Gustin, W., *Quasi-convex sets*, Canad. J. Math. 2(1950n), 489–507.
9. Korneichuk, N. P., *Extremal Problems of Approximation Theory* (in Russian), Nauka, Moscow 1976.
10. Korneichuk, N. P., Ligun, A. A. and Doronin, V. G., *Approximation with Restrictions* (in Ruîsian) Naukova Dumka, Kiew 1982.
11 Muntean, I., *Support points of p-convex sets*, Proc. Colloq. Approx. and Optimization (Cluj-Napoca 1984).
12. Muntean, I., *A multiplier rule in p-convex programming*, Seminar on Math. Anal., Univ. of Cluj-Napoca, Preprint Nr. 7 (1985) pp. 149–156.
13. Nikolski, V. N., *Extension of a theorem of A. N. Kolmogorov to Banach spaces* (in Russian), Studies on the Contemporary Problems of Constructive Function Theory (red. V. I. Smirnov), Moscow 1961, pp. 335–337.
14. Nikolski, V. N., *Best approximation by elements of convex sets in normed linear spaces* (in Russian) Uch. Zap. Kalinin Gos. Ped. Inst. **29**(1963), 85–119.
15. Ponstein, J., *Seven kinds of convexity*, SIAM Rev. **9**(1967), 115–119.
16. Rubinshtein, G. Sh., *On an extremal problem in a normed linear space* (în Russian), Sibirskii Mat. J. **6**(1965), 711–714.
17. Singer, I., *Choquet spaces and best approximation*, Math. Ann. **148**(1962), 330–340.
18. Singer, I., *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*. Springer-Verlag, Berlin — Heidelberg — New York and Editura Academiei Bucharest, 1970.

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