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# RELATIONS BETWEEN THE HOMOMORPHISMS OF $(n+1)$-GROUPS AND THE HOMOMORPHISMS OF THEIR EXTENSIONS AND REDUCES

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1. M. Hosszú [4] has proved that $(G, \varphi)$ is an $(n+1)$-group iff there is a binary operation "." defined on the same set $G$, a certain automorphism $\alpha : G \to G$ of $(G,.)$ and an element $a \in G$ such that

(1.1) $(G.,)$ is a group;

(1.2) $\alpha(a) = a$ and $\alpha^n$ is an inner automorphism of $(G.,)$ defined by $a$, i.e. $\alpha^n(x) = a\,x\,a^{-1}$ for any $x \in G$;

(1.3) $\varphi(x_1, x_2, \ldots, x_{n+1}) = x_1\,\alpha(x_2) \ldots \alpha^n(x_{n+1})$ a for any $x_1$ $x_2$ $\ldots$, $x_{n+1} \in G$.

If $(G, .\,\varphi, \alpha, a)$ satisfies the Hosszú conditions, then the $n+1$-group $(G, \varphi)$ is denoted by $\mathrm{Ext}^n_{\alpha,a}(G.,)$ and it is called the $n+1$-ary extension of the group $(G.,)$ with respect to $(\alpha, a)$ and $(G,., \alpha, a)$ is called a reduce of the $(G, \varphi)$.

W.A. Dudek and I. Michalski [3] have generalized the Hoszú theorem such: An $n+1$-groupoid $(G, \Phi)$ is an $n+1$-group if and only if $\Phi$ is of the form

(1.4) $\Phi(x_1, \ldots, x_n, x_{n+1}) = \varphi_{s+1}(x_1, \gamma(x_2), \ldots, \gamma^n(x_{n+1}), C_1^m)$, $n = ms$

where $(G, \varphi)$ is an $m+1$-group and $\gamma$ an automorphism of $(G, \varphi)$ such that.

(1.5) $\gamma(C_i) = C_i, \quad i = 1, 2, \ldots, m$

(1.6) $\varphi(\gamma^n(x), C_1^m) = \varphi(C_1^m, x), \quad \forall x \in G.$

The $n+1$-group $(G, \Phi)$ is denoted by $\mathrm{Ext}^s_\gamma c_1, \ldots, c_m(G, \varphi)$ and it is called the $n+1$-ary extension of the $m+1$-group $(G, \varphi)$ with respect to $(\gamma, c_1, \ldots, c_m)$.

Note that $\mathrm{Ext}^s_\gamma, c_1^m(G, \varphi)$ is the same with $\mathrm{Der}^s_\gamma, c_1^m\,(G, \varphi)$ from [3].

We denoted an $m+1$-group by $\overset{m+1}{(G, \varphi)}$.

In an $m+1$-group $(G, \varphi)$ to every $e \in G$ there exists a unique skew element $e_\varphi \in G$ such that

$$\varphi(\overset{(i-1)}{e}, e_\varphi, \overset{(m-i)}{e}, x) = \varphi(x, \overset{(i-1)}{e}, e_\varphi, \overset{(m-i)}{e}) = x$$

for any $x \in G, i = 2, \ldots, m$.

In [2] we have proved the following :

THEOREM 1. *If* $(G, \varphi)$ *is an* $m+1$*-group then* $(G, \Phi) = Ext^{s}_{\gamma, c_1^m}(G, \varphi)$ *iff there exists on G the same structure of group* $(G,.)$ *and exist* $a, b \in G$, $\alpha$, $\beta \in Aut(G,.)$ *such that.*

(1.7) $(G, \varphi) = \mathrm{Ext}^{m}_{\alpha,a}(G,.)$, $(G, \Phi) = \mathrm{Ext}^{n}_{\beta,b}(G,\cdot)$

*and*

(1.8) $\alpha \circ \beta = \beta \cdot \alpha$, $\alpha(b) = b$, $\beta(a) = a$.

Let $(G, \varphi)$ and $(G', \psi)$ be two $m+1$-groups and $(G,., \alpha, a)$ and $(G',., \alpha', a')$ some of theirs reduces. In the sequel we will need the following result (see [1]).

THEOREM 2. *A map* $f : G \to G'$ *is a homomorphism of* $m+1$*-group s iff it exists a homomorphism of binary group* $g : G \to G'$ *such that*

(1.9) $f(x) = f(1)\, g(x)$

(1.10) $g(\alpha(x))\alpha'(f(1)) = \alpha'(f(1))\alpha'(g(x))$

(1.11) $f(a) = \psi(f(1), f(1), \ldots, f(1)$

*where 1 is the unit element of* $(G,.)$.

2. Our aim is to give sufficient conditions in order a homomorphism of $m+1$-groups to be a homomorphism of their $n+1$-extensions and conversely. Then we will establish some relations between some homomorphisms of polyadic groups simultaneous reducible.

Let $(G, \Phi) = \mathrm{Ext}^{s}_{\gamma, c_1^m}(G, \varphi)$ be and $(G', \Psi) = \mathrm{Ext}^{s}_{\gamma', c_1'^m}(G', \psi)$.

*Remark.* From Theorem 1 follows that there exists on $G$ the same binary group operation, the elements $a$, $b \in G$ and $\alpha, \beta \in \mathrm{Aut}(G,.)$ such that (1.7) and (1.8) are verified. Again from same theorem it follows that we can choose for the reduction of $(G', \psi)$ the element $e' = f(e)$ which will be the unit element in $(G',.)$.

The pairs of polyadic groups $\overset{m+1}{(G, \varphi)}$ and $\overset{n+1}{(G, \Phi)}$; $\overset{m+1}{(G', \varphi)}$ and $\overset{n+1}{(G', \psi)}$ are reduce to the same binary operation by

(2.1), $x \cdot y = \varphi(x, \overset{m-2}{e}, e_{\varphi}, y)$ and $u.v = \psi(u, \overset{m-2}{e'}, e'_{\Psi}, v)$ respectively. Therefore, we have (1.7), (1.8) and

$$(G', \psi) = \mathrm{Ext}^{m}_{\alpha', a'}(G',.) \quad (G', \Psi) = \mathrm{Ext}^{n}_{\beta', b'}(G',.)$$

where

$$a = \varphi(\overset{m+1}{e}), \quad a' = \psi(\overset{m+1}{e'})$$

(2.2) $b = \Phi(\overset{n+1}{e})$, $b' = \Psi(\overset{n+1}{e'})$

and

(2.3) $\gamma \circ \alpha = \alpha \circ \gamma = \beta$; $\gamma \circ \alpha' = \alpha' \circ \gamma' = \beta'$



Since $f(e)$ is the unit of $(G',.)$, from Theorem 1 we obtain

COROLLARY 1. *The mapping* $f : (G, \varphi) \to (G'. \psi)$ *is a homomorphism of* $m+1$*-goups iff f is a homomorphism of binary groups and following relations are verified*

(2.4) $f \circ \alpha = \alpha' \circ f$,

(2.5) $f(\varphi(\overset{m+1}{e})) = \Psi(\overset{m+1}{e'})$.

COROLLARY 2. *The mapping* $f : (G, \Phi) \to (G', \Psi)$ *is a homomorphism of* $n+1$*-groups iff f is a homomorphism of binary groups and we have*

(2.6) $f \circ \beta = \beta' \circ f$,

(2.7) $f(\Phi(\overset{n+1}{e})) = \psi(\overset{n+1}{e'})$

THEOREM 1. *If* $f : (G, \varphi) \to (G', \psi)$ *is a homomorphism of* $m+1$*-groups and the following conditions are verified*

(2.8) $f \circ \gamma = \gamma' \circ f$

(2.9) $\exists\, e \in G$ *such that* $f(\varphi(e, c_1, \ldots, c_m)) = \psi(f(e), c'_1, \ldots, c'_m)$ *then f is a homomorphism of* $n+1$*-groups* $(G, \Phi)$ *and* $(G', \Psi)$.

*Proof.* We define on $G$ a binary operation by (2.1) and we can define on $G'$ a binary operation, too, such that $e' = f(e)$ is the identical element of $(G',.)$. Then $b' = \Psi(f(e), \ldots, f(e))$.

It is well known (see [2]) that

(2.10) $b = \varphi(e, c_1, \ldots, c_m)\, a^{s+1}$ and $b' = \psi(e', c'_1, \ldots, c'_m)\, a'^{s+1}$

Now we will prove that corollary 2 is true.

Since $f(e)$ is the unit element in $(G', .)$ from (1.9) we obtain that $f$ is a homomorphism of binary groups, and from (1.10) we have

$$f \circ \alpha = \alpha' \circ f.$$

From this equality, (2.3) and (2.8) we can write

$$f \circ \beta = f \circ \gamma \circ \alpha = \gamma' \circ f \circ \alpha = \gamma' \circ \alpha \circ f = \beta' \circ f$$

Hence (2.6) yield.

Now, we will show that (2.7) is true. Indeed, from our hypotheses it is sufficient to prove that

$$f(b) = \Psi(f(e), f(e), \ldots, f(e))$$

since $f(e) = e'$ is the unit element in $(G',.)$ and using (1.4) and (1.3), we obtain

$$\Psi(f(e), f(e), \ldots, f(e)) = \psi_{s+1}(f(e), \gamma'(f(e), \ldots, \gamma'^{n} f(e), c'_1, \ldots$$
$$\ldots, c'_m)) = \psi(f(e), c'_1, c'_2, \ldots, c'_m)a'^{s+1}$$

Because $f$ is homomorphism of binary groups, from (2.10), (2.5) and (2.9) we have

$$f(b) = f(\varphi(e, c_1, \ldots, c_m)a^{s+1}) = f(\varphi(e, c_1, \ldots, c_m))f(a^{s+1}) =$$
$$= \psi(f(e), c_1', \ldots, c_m')a'^{s+1}$$

Using (2.10) we obtain $f(b) = b'$. Therefore (2.7) yield and from Corollary 2 we have that $f$ is a homomorphism of $n+1$-groups $(G, \Phi)$ and $(G', \Psi)$.

THEOREM II. *If $f: (G, \Phi) \to (G', \Psi)$ is a homomorphism of $n+1$-groups and relation (2.8) holds and there exists $e \in G$ such that*

$$f(\varphi(e, \ldots, e)) = \psi(f(e), \ldots, f(e))$$

*then $f$ is a homomorphism of $m+1$-groups.*

*Proof.* We do same reduction as in Theorem I. Since $f$ is a homomorphism of $n+1$-groups it follows from Theorem 2 that $f$ is a homomorphism of binary groups and relations (2.6) and (2.7) are verified. Using Corollary 1 it is sufficient to show that (2.4) is true.

Indeed, from (1.8) and (2.8) we have

$$f\circ\alpha = f\circ\gamma_{\circ}^{-1}\circ\beta = \gamma'^{-1}\circ f\circ\beta = \gamma'^{-1}\circ\beta'\circ f = \alpha'\circ f$$

Thus proofs of Theorem 2 is complete.

From Theorem 1 follows Lemma 4 from [3].

COROLLARY 3. *If $f: G \to G'$ is a homomorphism of $m+1$-groups and relation (2.8) and $f(C_i) = C_i'$, $i = 1, 2, \ldots, n$ are true then $f$ is a homomorphism of $n+1$-groups.*

3. DEFINITION 1. The polyadic groups $(G, \varphi)$, $(G, \Phi)$ are called simultaneous reducible if there exists a same binary group operation on $G$ such that

$$(G, \varphi) \in \mathrm{Ext}\,(G, \cdot) \quad \text{and} \quad (G, \Phi) \in \mathrm{Ext}(G, \cdot)$$

It is known that the polyadic groups $(\overset{m+1}{G}, \varphi)$ and $(\overset{n+1}{G}, \Phi)$ are simultaneous reducible iff there exists an element $e \in G$ such that

$$(3.1) \quad \varphi(x, \overset{m-2}{e}, e_\varphi, y) = \Phi(x, \overset{n-2}{e}, e_\Phi, y), \qquad \forall x, y \in G.$$

Suppose that the polyadic groups $(G, \varphi)$ and $(G, \Phi)$ are simultaneous reducible and $(G', \psi)$ and $(G', \Psi)$ are simultaneous reducible too and relation (3.1) is verified.

It is known that if the polyadic groups $(\overset{m+1}{G'}, \psi)$ and $(\overset{n+1}{G'}, \Psi)$ verify an analogous relation with (3.1) then we have

$$(3.2) \quad \psi(x, \overset{m-2}{z}, z_\Psi, y) = \Psi(x, \overset{n-2}{z}, z_\Psi, y) \qquad \forall x, y, z \in G'$$



We can choose in (3.2) $z = f(e) = e'$.

From our hypothesis we can write that

$$(\overset{n+1}{G}, \Phi) = \mathrm{Ext}^s_{\substack{\gamma m-1 \\ e,v}}(G, \varphi) \quad \text{and} \quad (G', \Psi) = \mathrm{Ext}^s_{\substack{\gamma', m-1 \\ e'v'}}(G, \psi)$$

where

$$v = \varphi(\overset{m-s}{e}, \overset{s+1}{e_\varphi}); \quad \gamma = \Phi(e, \varphi(e_\varphi, \overset{m-2}{e}, x, e), \overset{n-2}{e}, e_\Phi)$$

and

$$v' = \psi(\overset{n-s}{e'}, \overset{s+1}{e_\varphi}); \quad \gamma' = \Psi(e', \psi(e_\Psi, \overset{m-2}{e'}, x, e'), \overset{n-2}{e'}\, e_\Psi).$$

COROLLARY 4. *If the mapping $f: G \to G'$ is a homomorphism of $m+1$-groups and the relation*

$$f\circ\gamma = \gamma'\circ f$$

*is verified, then $f$ is homomorphism of $n+1$-groups $(G, \Phi)$ and $(G', \Psi)$.*

From Theorem 1 is sufficient to show that (2.9) is true. Relation (2.9) becomes

$$\psi(e', \ldots, e', f(v)) = \psi(e', \ldots, e', v')$$

Therefore $f(v) = v'$ or $\psi(\overset{m-s}{e'}, f(\overset{s+1}{e_\varphi})) = \psi(\overset{m-s}{e'}, \overset{s+1}{e'_\Psi})$

But $f(e_\varphi) = e'_\Psi$. Indeed, since $f$ is a homomorphism of $m+1$-groups we obtain $f(\varphi(x, e, \ldots, e, e_\varphi)) = f(x) = \psi(f(x)e', \ldots$

$$\ldots, e'\, f(e_\varphi)) = \psi(f(x), e', \ldots, e', e'_\psi).$$

Hence $f(e_\varphi) = e'_\psi$. This completes the proof.

REFERENCES


1. I. Corovei and Purdea I., *Relations between the homomorphisms of n-groups and the homomorphisms of their reduces*, Math., 25(48) (1983), 131–135
2. Corovei I. and Purdea I. *The reduction of an $m+1$-group and its extensions to the same group.* "Babeş-Bolyai" University Fac. Math. Phys. Seminar of Algebra Preprint no. 5, 1988, pp. 63–68.
3. Dudek, W. A., Michalski J., *On a generalizations of Hosszú Theorem Demonstratic* Math. V XV (1982), 783–805
4. Hosszú V., *On the explicit form of n-group operations.* Bibl. Math. Debrecen 10 (1963), 88–92.


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