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ON SOME INEQUALITIES FOR CONVEX - DOMINATED FUNCTIONS

SEVER S. DRAGOMIR and NICOLETA M. IONESCU
(Băile Herculane)

Abstract. In this paper we shall give some inequalities for convex-dominated functions which improve the well-known results of Jensen, Fuchs, Jensen — Steffensen, Pečarić, Barlow — Marshall—Proschan and Vasić—Mijalković.

We shall introduce the following class of functions.

DEFINITION 1. Let $g : I \to \mathbb{R}$ be a given convex function on interval I from $\mathbb{R}$. The real function $f : I \to R$ is called g-convex-dominated on I if the following condition is satisfied :

$$|\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y)| \leqslant \leqslant \lambda g(x) + (1-\lambda)\, g(y) - g(\lambda x + (1-\lambda) y) \tag{1}$$

for all x, y in I and $\lambda \in [0, 1]$.

The next simple characterization of convex-dominated functions is valid.

LEMMA 1. *Let g be a convex function on I and $f : I \to \mathbb{R}$. Then the following statements are equivalent :*

(i) *f is g-convex-dominated on I ;*

(ii) *$g - f$ and $g + f$ are convex on I ;*

(iii) *there exists two convex mappings h, l on I such that $f = 1/2(h - l)$ and $g = 1/2(h + l)$.*

Proof. "(i) ⇔ (ii)". Condition (1) is equivalent to

$$\lambda(g(x) - f(x)) + (1-\lambda)\,(g(y) - g(y)) \geqslant g(\lambda x + (1-\lambda) y) - f(\lambda x + (1-\lambda) y)$$

and

$$\lambda(g(x) + f(x)) + (1-\lambda)(g(y) + f(y)) \geqslant g(\lambda x + (1-\lambda) y) + f(\lambda x + (1-\lambda) y)$$

for all x, y in I and $\lambda \in [0, 1]$, i.e., $g - f$ and $g + f$ are convex on I iff (1) holds.

"(ii) ⇔ (iii)". It's obvious.

Now, let $F(I)$ be the linear space of all real valued functions defined on I and $J : F(I) \to \mathbb{R}$ be a functional satisfying the properties:

(J1) $J(\alpha f + \beta g) = \alpha J(f) + \beta J(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in F(I)$;

(J 2) $J(f) \geqslant 0$ for all convex function f on I.

The following lemma plays a very important role in the sequel.

LEMMA 2. *Let J be a functional satisfying conditions (J1), (J2). Then for every convex function g and for every g-convex-dominated function f on I, the following inequality holds:*

$$|J(f)| \leqslant J(g). \tag{2}$$

Proof. Let g be a convex function and f be g-convex-dominated on I. By Lemma 1 it follows that $g - f$ and $g + f$ are convex on I. Then

$$0 \leqslant J(g - f) = J(g) - J(f) \text{ and } 0 \leqslant J(g + f) = J(g) + J(f)$$

which gives

$$-J(g) \leqslant J(f) \leqslant J(g).$$

Since $J(g) \geqslant 0$, inequality (2) is proven.

COROLLARY 2.1. *Let $f \in C^2[a, b]$, $h \in C[a, b]$, $h \geqslant 0$ and*

$$|f''(t)| \leqslant g(t) \text{ for all } t \in [a, b].$$

Then for all functional J having the properties (J1), (J2), the following inequalities hold:

$$|J(f)| \leqslant J\left(\int_a^{\cdot}\left(\int_a^t h(s)\, ds\right) dt\right) \tag{3}$$

and

$$|J(f)| \leqslant J\left(\int_a^{\cdot}\left(\int_a^t |f''(s)|\, ds\right) dt\right). \tag{4}$$

COROLLARY 2.2. *Let f, J be as above and $M := \sup_{t\in[a,b]} |f''(t)|$*

Then the following inequality is valid:

$$|J(f) \leqslant 1/2\, MJ(e^2) \tag{5}$$

where $e(x) = x$ on the interval $[a, b]$.

The above corollaries follow by Lemma 2 observing that:

$\int_a^{\cdot}\left(\int_a^t h(s)\, ds\right) dt$ is convex, f is $\int_a^{\cdot}\left(\int_a^t h(s)\, ds\right) dt$ — convex-dominated; $\int_a^{\cdot}\left(\int_a^t |f''(t)|\, ds\right) dt$ is convex, f is $\int_a^{\cdot}\left(\int_a^t |f''(s)|\, ds\right) dt$ — convex —

dominated and $1/2\, Me^2$ is convex and f is $1/2\, Me^2$ — convex dominated on $[a, b]$.

The following improvement of Jensen inequality holds.

THEOREM 1. *Let g be a given convex function on I and $f : I \to R$ be g-convex-dominated. Then for every $x_i \in I$, $p_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ such that $P_n := \sum_{i=1}^n p_i > 0$, we have the inequality:*

$$\left|\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\right| \leqslant \frac{1}{P_n}\sum_{i=1}^n p_i g(x_i) - g\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right). \tag{6}$$

Proof. Let us consider the functional:

$$J(f) := \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right),\ f \in F(I).$$

Then J satisfy conditions (J1) and (J2) (by Jensen's inequality). Applying Lemma 2, we obtain inequality (6).

The proof is finished.

Remark 1°. Let f, h be as in Corollary 2.1. Then we can put in (6) $g = H$ or $g = F$ where

$$H(x) := \int_a^x\left(\int_a^t h(s)\, ds\right) dt,\ F(x) := \int_a^x\left(\int_a^t |f''(s)|\, ds\right) dt,\ x \in [a, b].$$

2°. If $f \in C^2\, [a, b]$ and $M := \sup_{t\in[a,b]} |f''(t)|$, then the following inequality is valid:

$$\left|f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)\right| \leqslant \frac{M}{2}\,\frac{P_n \sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i\right)^2}{P_n^2} \tag{7}$$

where $x_i \in [a, b]$ and p_i $(1 \leqslant i \leqslant n)$ are as above (see also Theorem 1 from [1]).

Now, we shall give an improvement of Fuchs generalization of the Majorization theorem (see [3]). This result can be written in the following form:

THEOREM 2. *Let $a_1 \geqslant \ldots \geqslant a_s$, $b_1 \geqslant \ldots \geqslant b_s$ and $q_1, \ldots, q_s$ be real numbers such that:*

$$\sum_{i=1}^k q_i a_i \leqslant \sum_{i=1}^k q_i b_i\ (1 \leqslant k \leqslant s - 1),\ \sum_{i=1}^s q_i a_i = \sum_{i=1}^s q_i b_i.$$

If g is convex on I and f is g-convex-dominated on I, then the following inequality holds:

$$\left|\sum_{i=1}^s q_i(f(b_i) - f(a_i))\right| \leqslant \sum_{i=1}^s (g(b_i) - g(a_i)). \tag{8}$$

Proof. Let consider the functional :

$$J(f) := \sum_{i=1}^{s} q_i(f(b_i) - f(a_i)),\ f \in F(I).$$

Then J satisfies conditions (J1) and (J2) (by Fuchs' inequality see also Theorem B from [4]). Applying Lemma 2, we deduce inequality (8).

Remarks 3°. Let f, g, h, H, F be as in Remark 1°, then in (8) we can put $g = H$ or $g = F$.

4°. Let $f \in C^2[a, b]$ and $M := \sup_{t\in[a,b]} |f''(t)|$, then the following inequality holds :

$$\left| \sum_{i=1}^{s} q_i(f(b_i) - f(a_i)) \right| \leqslant M/2 \sum_{i=1}^{s} q_i(b_i^2 - a_i^2), \tag{9}$$

where a_i, b_i, q_i $(1 \leqslant i \leqslant s)$ are as above.

Now, we shall give an improvement of Jensen-Steffensen inequality.

THEOREM 3. *Let x and p be two n-tuples of real numbers such that $x_i \in I$ $(1 \leqslant i \leqslant n$, I is an interval from $\mathbb{R})$ and $P_n > 0$. Then the following sentences are equivalent :*

(i) *For every convex function $g : I \to \mathbb{R}$, for every g-convex-dominated function f and for all monotonic n-tuple x the inequality (6) holds ;*

(ii) $0 \leqslant P_n \leqslant P_n$ *for all* $k = 1, 2, \ldots, n-1$.

Proof. "(i) $\Rightarrow$ (ii)". It's obvious by Jensen-Steffensen theorem.

"(ii) $\Rightarrow$ (i)". Let us consider the functional :

$$J(f) := \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right),\ f \in F(I).$$

Then J verifies conditions (J1) and (J2) (by Jensen-Steffensen inequality; see for example [4], Theorem A). Aplying Lemma 2, we obtain (6).

Remarks 1° and 2° are also valid if p_i $(1 \leqslant i \leqslant n)$ satisfies condition (ii) of the above theorem.

Now, we shall give another result which improves Pečarić's theorem (see [4], Theorem 1) :

THEOREM 4. *Let x be nonincreasing n-tuple of real numbers, $x_i \in I$ $(1 \leqslant i \leqslant n)$, p real n-tuple and exists $j \in (1, 2, \ldots, n)$ such that :*

$$\sum_{i=1}^{k} p_i(x_i - x_j) \leqslant 0 \text{ for every } k \text{ such that } x_k \geqslant \bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$$

(10)

$$\sum_{i=k}^{n} p_i(x_i - x_j) \geqslant 0 \text{ for every } k \text{ such that } x_k \leqslant \bar{x},$$

(if $x_1 \leqslant \bar{x}$ the first condition in (10) is taken to be vacuous, if $x_n \geqslant \bar{x}$ the second condition in (10) is taken to be vacuous). If $\bar{x} \in I$, then for every

convex function $g : I \to \mathbb{R}$ and for every g-convex dominated function $f : I \to \mathbb{R}$, we have :

$$g\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i g(x_i) \geqslant \left| f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n} p_i f(x_i)\right|. \tag{11}$$

If the inverse inequalities in (10) hold, then (6) holds.

The proof follows by a similar argument to that in the proof of the previous theorem using the result of J. E. Pečarić([4], Theorem 1). We omit the details.

By Theorem 2 from [4] we also obtain :

THEOREM 5. *Let x and p be two n-tuple of real numbers such that $x_i \in I$ $(1 \leqslant i \leqslant n)$, $\bar{x} \in I$, $P_n \geqslant 0$. Then the following sentences are equivalent :*

(i) *inequality (11) holds for every convex function $g : I \to \mathbb{R}$, for every g-convex-dominated function $f : I \to R$ and for all monotonic n-tuple x ;*

(ii) *there exists $m \in (1, 2, \ldots, n)$ such that $P_k \leqslant 0$ $(k < m)$*

$$\text{and } \bar{P}_k \leqslant 0\ (k > m), \text{ where } \bar{P}_k : P_n - P_{k-1}. \tag{12}$$

Remark 5°. Let f be as in Corollary 2.2. If p, x satisfy conditions (10) or x is a monotonic n-tuple and p verifies (12), then the following inequality holds :

$$M/2\left[\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2 - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2\right] \geqslant \left| f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)\right|. \tag{13}$$

Now, we shall give an improvement of Barlow-Marshall-Proschan inequality.

THEOREM 6. *Let $x_1 \leqslant \ldots \leqslant x_m \leqslant 0 \leqslant x_{m+1} \leqslant \ldots \leqslant x_n (m \in (1, \ldots \ldots, n))$, $x_i \in I$ $(1 \leqslant i \leqslant n$, $0 \in I)$ and p is real n-tuple.*

(i) Inequality

$$\sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i\right) - \left(\sum_{i=1}^{n} p_i - 1\right) g(0) \geqslant \left|\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) - \left(\sum_{i=1}^{n} p_i - 1\right) f(0)\right| \tag{14}$$

holds for every convex function $g : I \to \mathbb{R}$ and for every g-convex-dominated function $f : I \to \mathbb{R}$ if and only if

$$0 \leqslant P_k \leqslant 1\ (1 \leqslant k \leqslant m);\ 0 \leqslant \bar{P}_k \leqslant 1\ (m+1 \leqslant k \leqslant n). \tag{15}$$

(ii) Let $\sum_{i=1}^{n} p_i x_i \in I$. Then following the ineuality holds

$$\left(\sum_{i=1}^{n} p_i - 1\right) g(0) + g\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i g(x_i) \geqslant \left|\left(\sum_{i=1}^{n} p_i - 1\right) f(0) + f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i)\right| \tag{14'}$$

if and only if there exists $j \leqslant m$ *such that*

(16) $\quad P_i \leqslant 0 (i < j)$; $P_i \geqslant 1 (j \leqslant i \leqslant m)$; $\bar{P}_i \leqslant 0$ $(i \geqslant m+1)$;

or exists $j \geqslant m$ *such that*:

(17) $\quad P_i \leqslant 0 (i \leqslant m)$; $\bar{P}_i \geqslant 1 (j \geqslant i \geqslant m+1)$; $\bar{P}_i \leqslant 0 (i > j)$.

The proof follows by Theorem of Barlow-Marshall-Proschan (see [2] or [4] Corollary 1) and by Lemma 2 for the functional

$$J(f) := \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) - \left(\sum_{i=1}^{n} p_i - 1\right) f(0).$$

We omit the details.

Remark 6°. Let $f \in C^2[a, b]$, $M = \sup_{t \in [a,b]} |f''(t)|$ and $x_i \in I$ $(1 \leqslant i \leqslant n)$ be as above. If p_i $(1 \leqslant i \leqslant n)$ verifies (15) we have:

$$M/2\left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \geqslant \left|\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) - \left(\sum_{i=1}^{n} p_i - 1\right) f(0)\right|. \tag{19}$$

Let $\sum_{i=1}^{n} p_i x_i \in I$ and p_i $(1 \leqslant i \leqslant n)$ satisfy (16) or (17), then

$$M/2\left[\left(\sum_{i=1}^{n} p_i x_i\right)^2 - \sum_{i=1}^{n} p_i x_i^2\right] \geqslant \left|\left(\sum_{i=1}^{n} p_i - 1\right) f(0) + f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i)\right|. \tag{19'}$$

Now, let H be a finite nonempty set of positive integers. If $p_i > 0$, $x_i \in [a, b]$ and f is a real function defined on $[a, b]$, let us denote:

$$F(H, f) := \left(\sum_{i \in H} p_i\right) f\left(\frac{\sum_{i \in H} p_i x_i}{\sum_{i \in H} p_i}\right) - \sum_{i \in H} p_i f(x_i).$$

P.M. Vasić and Z. Mijalković have proved in [5] that if H, L are finite nonempty set of positive integers, $H \cap L \neq \emptyset$, $p_k > 0$, $x_k \in [a, b]$, $k \in H \cup L$ and f is convex on $[a, b]$, then

$$F(H \cup L, f) \geqslant F(H, f) + F(L, f). \tag{20}$$

We give the following improvement of this fact.

THEOREM 7. *Let g be a given convex function on $[a, b]$ and $f: [a, b] \to \mathbb{R}$ be g-convex-dominated. Then for every $x_k \in [a, b]$, $p_k > 0$ $(k \in H \cup L)$, we have the inequality*:

$$F(H \cup L, g) - F(H, g) - F(L, g) \geqslant |F(H \cup L, f) - F(H, f) - F(L, f)|. \tag{21}$$

The proof follows by inequality (20) and by Lemma 2.

Remark 7°. If $f \in C^2[a, b]$, $M = \sup_{t \in [a,b]} |f''(t)|$, $\delta_{H,L}(x, p) := \frac{P_H P_L}{P_{H \cup L}} (A_H(x, p) - A_L(x, p))^2$ where $P_H := \sum_{i \in H} p_i$ and $A_H := \frac{1}{P_H} \sum_{i \in H} p_i x_i$ we obtain the inequality:

$$|F(H \cup L, f) - F(H, f) - F(L, f)| \leqslant M/2\ \delta_{H,L}(x, p) \tag{22}$$

(see also [1], Theorem 2).

Remark 8°. If in Remarks 2°, 4°, 5°, 7° we consider $[a, b] \subset (0, \infty)$, $f(t) := \ln t$, $M = 1/a^2$ or in Remarks 2° – 7°, we put $f(t) = \exp t$, $M = \exp b$ or $[a, b] \subset (0, \infty)$ and $f(t) := t^c$, $c \in [2, \infty)$, $M = c(c-1) b^{c-2}$ we can obtain some interesting inequalities for real numbers (see also [1]). We omit the details.

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Şcoala Generală Băile Herculane,
1600 Băile Herculane
Jud. Caraş-Severin

Şcoala Generală Mehadia
1612 Mehadia
Jud. Caraş-Severin