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A NOTE ON THE JENSEN-HADAMARD INEQUALITY

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1. J.L.W.V. Jensen [4], [5] was the first mathematician who discovered the great importance and perspective of convex functions. Among many important general inequalities (See also [9], [10]) his results contain as a special case the following relations:

$$(1)\qquad (b-a)\,f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)\,dx \leqslant (b-a)\frac{f(a)+f(b)}{2}$$

for a continuous and convex function $f:\ [a, b] \to \mathbb{R}$.

Inequality (1) was proved independently by J. Hadamard [2], under a slightly stronger condition: by supposing that f has an increasing derivative on $[a, b]$. We note that the left side of (1) is called sometimes as the "Hadamard inequality", while the right side as the "Jensen inequality" (Or vice-versa). There are also some papers which attribute inequality (1) completely to J. Hadamard. In our opinion, a more penetrating study on the history and priority related to convex functions justifies to call (1) as the "Jensen — Hadamard inequality".

In [3] I. B. Lacković has obtained the following generalization of (1):

$$(2)\qquad f\left(\frac{1}{2n}\sum_{k=1}^{2n} a_k\right) \leqslant \frac{1}{n}\left(\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx + \ldots + \frac{1}{a_{2n}-a_{2n-1}}\int_{a_{2n-1}}^{a_{2n}} f(x)\,dx\right) \leqslant \frac{1}{2n}\sum_{k=1}^{2n} f(a_k),$$

where $f:\ [a, b] \to \mathbb{R}$ has an increasing derivative on $[a, b]$ and $a_i \in [a, b]$ $(i = 1, 2, \ldots, 2n)$ with $a_1 < a_2 < \ldots < a_{2n}$.
We notice that, in fact (2) is valid for f continuous and convex, since the proof is based essentially on (1).

A very general extension for the right-hand side of (1) was proved by A. Lupaş [8]. Let $C[a, b]$ denote the normed linear space of all functions $f: [a, b] \to \mathbb{R}$ which are continuous on $[a, b]$ and let $F: C[a, b] \to \mathbb{R}$ be a positive linear functional with $F(e_0) = 1$ (where $e_0(t) = 1$, $t \in [a, b]$). If $f \in C[a, b]$ is a convex function on $[a, b]$ then $x_0 \in [a, b]$, where $x_0 = F(e_1)$ (with $e_1(t) = t$, $t \in [a, b]$) and

$$f(x_0) \leqslant F(f). \tag{3}$$

When we consider the functionals of the form $F_1(f) = \sum_{j=1}^{n} w_j f(x_j)$ with $w_j \geqslant 0$, $F_1(e_0) = 1$ and one finds:

$$f\left(\sum_{j=1}^{n} w_j x_j\right) \leqslant \sum_{j=1}^{n} w_j f(x_j) \tag{4}$$

for $x_j \in [a, b]$, $w_j \geqslant 0$ $(j = 1, 2, \ldots, n)$, $\sum_{j=1}^{n} w_j = 1$. This is the classical Jensen inequality.

By considering $F_2(f) = \frac{1}{y - x} \int_x^y f(t)\, dt$, x, y being arbitrary distinct points from $[a, b]$, one obtains the right side of (1).

For the left-hand side of (1), one of us [11] has proved the following generalization, important in applications. Let $f: [a, b] \to \mathbb{R}$ be a $2k$-times differentiable function having continuous $2k$th derivative on $[a, b]$ and satisfying $f^{(2k)}(t) \geqslant 0$ $(>)$ for $t \in (a, b)$. Then one has the inequality:

$$\int_a^b f(x)\, dx \geqslant (>) \sum_{p=1}^{k} \frac{(b-a)^{2p-1}}{2^{2p-2} \cdot (2p-1)!} f^{(2p-2)}\left(\frac{a+b}{2}\right) \tag{5}$$

For other generalizations and new proofs, see e.g. [7], [14], [15]. For applications in analysis and number theory, see e.g. [2], [11], [12], [13].

2. In what follows, our aim is to prove a discrete analogous of (1) as well as to obtain some refinements of the Jensen–Hadamard inequality. At the end we will give an application of one of these refinements in the theory of Euler's Gamma function.

THEOREM 1. *Let $f: I \to \mathbb{R}$ ($I \subset \mathbb{R}$, interval) be a continuous convex function and let $a, b \in I$, $n \in N^*$. Then holds true the following inequality*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n+1}\, a + \left(1 - \frac{i}{n+1}\right) b\right) \leqslant \frac{f(a) + f(b)}{2} \tag{6}$$

Proof. Using relation (4) with $w_j = \frac{1}{n}$, $j = 1, 2, \ldots, n$, one finds successively:

$$f\left(\frac{\left(\sum_{i=1}^{n} m_i\right) a + \sum_{i=1}^{n} (1 - m_i)\, b}{n}\right) \leqslant \frac{1}{n} \sum_{i=1}^{n} f(m_i a + (1 - m_i) b) \leqslant$$

$$\leqslant \frac{\left(\sum_{i=1}^{n} m_i\right) f(a) + \sum_{i=1}^{n} (1 - m_i) f(b)}{n}$$

with $m_i > 0$ $(i = 1, 2, \ldots, n)$ arbitrary positive real numbers. Select $m_i = \frac{i}{n+1}$ and observe that $\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} (1 - m_i) = \frac{n}{2}$. This gives immediately (6).

For $n = 2$ we obtain:

COROLLARY 1. *If $f: I \to \mathbb{R}$ is continuous and convex, then for all $a, b \in I$ one has*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left[f\left(\frac{a+2b}{3}\right) + f\left(\frac{2a+b}{3}\right)\right] \leqslant \frac{f(a) + f(b)}{2} \tag{7}$$

For an application, choose $f(t) = -\log t$, $t > 0$. We get the following simple refinement of the arithmetic-geometric inequality:

$$\frac{a+b}{2} \geqslant \frac{1}{3}\sqrt{(a+2b)(2a+b)} \geqslant \sqrt{ab},\ a, b > 0 \tag{8}$$

For $f(t) = \frac{1}{t}$, $t > 0$ we have:

$$\frac{1}{A} \leqslant \frac{9(a+b)}{2(a+2b)(2a+b)} \leqslant \frac{1}{H} \tag{9}$$

where A and H denote the arithmetic and harmonic means of a, b, respectively.

The next result offers a refinement of the Jensen–Hadamard inequality.

THEOREM 2. *Let $f: I \to \mathbb{R}$ be a continuous convex function and $a, b \in I$ $(a < b)$, $n \in N^*$. Then one has the following inequalities:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\int_a^b \cdots \int_a^b f\left(\sum_{i=1}^{n+1} x_i/(n+1)\right) dx_1 \ldots dx_{n+1}}{(b-a)^{n+1}} \leqslant \tag{10}$$

$$\leqslant \frac{\int_a^b \cdots \int_a^b f\left(\sum_{i=1}^n x_i/n\right) dx_1 \ldots dx_n}{(b-a)^n} \leqslant \ldots \leqslant \frac{\int_a^b\int_a^b f((x_1+x_2)/2)\, dx_1\, dx_2}{(b-a)^2} \leqslant$$

$$\leqslant \frac{\int_a^b f(x)\, dx}{(b-a)} \leqslant \frac{f(a)+f(b)}{2}$$

Proof. By Jensen's inequality (See (4)) we can write:

$$\frac{1}{n+1}\left[f\left(\frac{x_1+\ldots+x_n}{n}\right)+f\left(\frac{x_2+\ldots+x_{n+1}}{n}\right)+\ldots+\right.$$

$$\left.+f\left(\frac{x_{n+1}+x_1+\ldots+x_{m-1}}{n}\right)\right] \geqslant$$

$$f\left(\frac{\frac{x_1+\ldots+x_n}{n}+\ldots+\frac{x_{n+1}+\ldots+x_{n-1}}{n}}{n+1}\right) = f\left(\frac{x+\ldots+x_{n+1}}{n+1}\right),$$

so by integration on $[a, b]^{n+1}$, we easily obtain

$$\int_a^b dt \int_a^b \cdots \int_a^b f\left(\frac{x_1+\ldots+x_n}{n}\right) dx_1 \ldots dx_n \geqslant$$

$$\geqslant \int_a^b \cdots \int_a^b f\left(\frac{x_1+\ldots+x_{n+1}}{n+1}\right) dx_1 \ldots dx_{n+1},$$

that is, the middle inequalities in (10).

On the other hand, it is well-known that for a continuous convex function f, one has $f(u)-f(v) \geqslant (u-v)f'_+(v)$, $u, v \in I$, where f'_+ denotes the right derivative of f. Choose $u=(x_1+\ldots+x_{n+1})/(n+1)$, $v=(a+b)/2$ and integrate the obtained inequality on $[a, b]^{n+1}$. Since

$$\int_a^b\int_a^b \cdots \int_a^b \frac{x_1+\ldots+x_{n+1}}{n+1} dx_1 \ldots dx_{n+1} = (b-a)^{n+1}\cdot\left(\frac{a+b}{2}\right),$$

we have obtained the first inequality of (10), which concludes the proof of Theorem 2. For n = 1 one gets:

COROLLARY 2. *If $f: I \to \mathbb{R}$ is continuous and convex, then for all $a, b \in I$, $a < b$ one has*

(11) $$f\left(\frac{a+b}{2}\right) \leqslant \frac{\int_a^b\int_a^b f\left(\frac{x+y}{2}\right) dx\, dy}{(b-a)^2} \leqslant \int_a^b \frac{f(x)\, dx}{(b-a)} \leqslant \frac{f(a)+f(b)}{2}$$

For an application, set $f(t) = -(\log\ \Gamma(t))' = -\psi(t)$, where Γ and ψ are the Euler gamma and digamma functions, respectively. It is well known ([1], [12], [16]) that $\psi''(t) < 0$ for $t > 0$, thus f is convex. Clearly, $\int_a^b \psi\left(\frac{x+y}{2}\right) dx = 2\log\Gamma\left(\frac{b+y}{2}\right) - 2\log\Gamma\left(\frac{a+y}{2}\right)$, so (11) may be written also in the form

(12) $$\psi\left(\frac{a+b}{2}\right) > \frac{4}{(b-a)^2}\left[\int_{\frac{a+b}{2}}^{a} \log\Gamma(t)\, dt - \int_a^{\frac{a+b}{2}} \log\Gamma(t)\, dt\right] >$$

$$> \log\frac{\Gamma(b)}{\Gamma(a)} > \frac{\psi(a)+\psi(b)}{2}.$$

For $a = x+s$, $b = x+1$ $(x > 0,\ 0 < s < 1)$ this contains a generalization and refinement of a result by D. Kershaw [6].

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