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THE DUAL OF THE DUAL IN MATHEMATICAL PROGRAMMING IN COMPLEX SPACE

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Abstract. The form and properties of the dual of the dual of a nonlinear programming problem in complex space are investigated.

1. Introduction. Let $A, B \in C^{m\times n}$; $c \in C^m$; $d \in C^n$. Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones.

The dual of the linear programming problem in complex space

(1) $\min \operatorname{Re}\langle z, d\rangle$ subject to $Az + B\bar{z} + c \in S$, $z \in T$,

is given by

(2) $\max \operatorname{Re}\langle u, -c\rangle$ subject to $-A^H u - B^T\bar{u} + d \in T^*$, $u \in S^*$.

Problem (2) can be written as

(3) $-\min \operatorname{Re}\langle u, c\rangle$ subject to $(-A^H)u + (-B^T)\bar{u} + d \in T^*$, $u \in S^*$.

Then the dual of (3), that is, the dual of the dual of (1), may be formed. The dual of the dual of (1) is exactly problem (1). Hence, the dual of the dual of a linear programming problem in complex space coincides with the primal problem. If the primal problem is not linear, does the dual of the dual coincide with the primal problem?

In this paper the form and properties of the dual of the dual of a nonlinear programming problem in the complex space are investigated. In the real space the form, properties and computational possibilities of the dual of the dual of a nonlinear programming problem are investigated by Unger and Hurter in [6].

2. Notation and Preliminaries. Let C^n (R^n) denote the n-dimensional complex (real) space and $C^{m\times n}(R^{m\times n})$ the set of $m\times n$ complex (real) matrices. If A is a matrix or a vector, then A^T, $\bar{A}$, A^H denote its transpose, complex conjugate and conjugate transpose respectively. For $z = (z_j) \in C^n$, $\operatorname{Re} z = (\operatorname{Re} z_j) \in R^n$ denotes the real part of z. For $z, v \in C^n$: $\langle z, v\rangle = v^H z$ denotes the inner product of z and v.

A nonempty set S in C^n is a *convex cone* if $S + S \subseteq S$ and if $r \in R$, $r > 0$, then $rS \subseteq S$.

For any nonempty set S in C^n let $S^* = \{v \in C^n : \operatorname{Re}\langle z, v\rangle \geqslant 0$ for all $z \in S\}$ the *polar* of S. If S is a nonempty set in C^n, then S^* is a closed convex cone. A nonempty set S in C^n is a closed convex cone if and only if $(S^*)^* = S$.

Let M be an open set in C^n and let $z^0 \in M$. (i) The function $f: M \to C$ is said to be *differentiable at* z^0 if there exist $2n$ complex numbers $A_1(z^0), \ldots, A_n(z^0), B_1(z^0), \ldots, B_n(z^0)$ and a function $h(\cdot\,; z^0): M \to C$ continuous at z^0 and vanishing at this point such that

$$f(z) - f(z^0) = \sum_{j=1}^{n} A_j(z^0)(z_j - z_j^0) + \sum_{j=1}^{n} B_j(z^0)(\bar{z}_j - \bar{z}_j^0) +$$

$$+ \| z - z^0 \| h(z;\, z^0) \quad \text{for all } z \in M.$$

If for $z = x + iy \in M (x, y \in R^n)$ we have $f(z) = u(x, y) + iv(x, y)$ then the function f is differentiable at $z^0 = x^0 + iy^0 \in M$ if and only if the functions u and v are differentiable at $(x^0, y^0) \in R^{2n}$. If we consider the formal differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \text{ and } \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for all $j \in \{1, \ldots, n\}$, we obtain that

$$A_j(z^0) = \frac{\partial f}{\partial z_j}(z^0), \quad B_j(z^0) = \frac{\partial f}{\partial \bar{z}_j}(z^0) \text{ for all } j \in \{1, \ldots, n\}.$$

If $f: M \to C$ is differentiable at $z^0 \in M$, then

$$\nabla_z f(z^0) = \left(\frac{\partial f}{\partial z_1}(z^0), \ldots, \frac{\partial f}{\partial z_n}(z^0)\right)^T \in C^n$$

$$\nabla_{\bar{z}} f(z^0) = \left(\frac{\partial f}{\partial \bar{z}_1}(z^0), \ldots, \frac{\partial f}{\partial \bar{z}_n}(z^0)\right)^T \in C^n.$$

The function $f: M \to C$ is said to be *differentiable* on M if it is differentiable at any $z^0 \in M$.

(ii) The function $f: M \to C$ is said to be *twice differentiable at* $z \in M$ if there exists a neighborhood V of z such that f is differentiable on V and the functions

$$\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial \bar{z}_1}, \ldots, \frac{\partial f}{\partial \bar{z}_n}: V \to C$$

are differentiable at z. If f is twice differentiable at z, then

$$\nabla^2_{zz} f(z) = \left(\frac{\partial}{\partial z_j}\left(\frac{\partial f}{\partial z_k}(z)\right)\right) \in C^{n \times n},$$

$$\nabla^2_{\bar{z}\bar{z}} f(z) = \left(\frac{\partial}{\partial \bar{z}_j}\left(\frac{\partial f}{\partial \bar{z}_k}(z)\right)\right) \in C^{n \times n},$$

$$\nabla^2_{\bar{z}z} f(z) = \left(\frac{\partial}{\partial \bar{z}_j}\left(\frac{\partial f}{\partial z_k}(z)\right)\right) \in C^{n \times n},$$

$$\nabla^2_{z\bar{z}} f(z) = \left(\frac{\partial}{\partial z_j}\left(\frac{\partial f}{\partial \bar{z}_k}(z)\right)\right) \in C^{n \times n}.$$

The function $f: M \to C$ is said to be *twice differentiable on* M if it is twice differentiable at any $z \in M$.

(iii) The function $g = (g_k): M \to C^m$ is said to be *differentiable at* $z \in M$ (on M) if for any $k \in \{1, \ldots, m\}$ the function g_k is differentiable at z (on M). If $g = (g_k): M \to C^m$ is differentiable at $z \in M$, then

$$\nabla_z g(z) = (\nabla_z g_1(z) \ldots \nabla_z g_m(z)) \in C^{n \times m}$$

$$\nabla_{\bar{z}} g(z) = (\nabla_{\bar{z}} g_1(z) \ldots \nabla_{\bar{z}} g_m(z)) \in C^{n \times m}.$$

If $g: M \to C^m$ is differentiable at $z \in M$, then

$$\nabla_z \bar{g}(z) = \overline{\nabla_{\bar{z}} g(z)}, \qquad \nabla_{\bar{z}} \bar{g}(z) = \overline{\nabla_z g(z)}.$$

(iv) The function $g = (g_k): M \to C^m$ is said to be *twice differentiable at* $z \in M$ (on M) if for any $k \in \{1, \ldots, m\}$ the function g_k is twice differentiable at z (on M).

(v) The function $g: M \to C^m$ is said to be *convex* with respect to closed convex cone $S \subseteq C^m$ if

(4) $$\lambda g(z) + (1 - \lambda)\, g(v) - g[\lambda z + (1 - \lambda) v] \in S,$$

for all $\lambda \in [0, 1]$ and $z, v \in M$.

If in addition g is differentiable on M, then from (4) it follows that

$$g(z) - g(v) - [\nabla_z g(v)]^T (z - v) - [\nabla_{\bar{z}} g(v)]^T (\bar{z} - \bar{v}) \in S$$

for all $z, v \in M$.

(vi) The function $g: M \to C^m$ is *concave* with respect to closed convex cone S if g is convex with respect to $-S$.

(vii) The function $f: M \to C$ is said to be *with convex real part* if it is convex with respect to the cone $\{v \in C : \operatorname{Re} v \geq 0\}$.

3. Background. Let M be a nonempty open set in C^n. Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ and $g: M \to C^m$ both be differentiable functions on M.

We shall consider the following primal problem:

(P) $$\inf \operatorname{Re} f(z) \text{ subject to } z \in M,\ g(z) \in S,\ z \in T.$$

The dual [1, 4] of this problem is given by

$$\sup \operatorname{Re}[f(z) - \langle (g(z), z), (u, v)\rangle]$$

subject to

(5)
$$\overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z (g(z), z)} \begin{pmatrix} u \\ v \end{pmatrix} - \nabla_{\bar z}(g(z), z) \overline{\begin{pmatrix} u \\ v \end{pmatrix}} = 0$$

$$(u, v) \in (S \times T)^*$$

Since $(S \times T)^* = S^* \times T^*$ and for any $z \in M$, $u \in C^m$, $v \in C^n$ we have

$$\nabla_z (g(z), z) = (\nabla_z g(z) \;\; I) \in C^{n \times (m+n)},$$

$$\nabla_{\bar z} (g(z), z) = (\nabla_{\bar z} g(z) \;\; 0) \in C^{n \times (m+n)},$$

$$\langle (g(z), z), (u, v)\rangle = \langle g(z), u\rangle + \langle z, v\rangle,$$

problem (5) can be written in the form

$$\sup \operatorname{Re}[f(z) - \langle g(z), u\rangle - \langle z, v\rangle]$$

subject to

(6)
$$\overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z g(z)}\, u - \nabla_{\bar z} g(z)\, \bar u - v = 0$$

$$u \in S^*, \quad v \in T^*.$$

We solve the equality constraints for v, substitute directly for v, and obtain the problem

$$\sup \operatorname{Re}[f(z) - \langle g(z), u\rangle - \langle z, \overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z g(z)} u - \nabla_{\bar z} g(z) u\rangle]$$

subject to

(D)
$$(z, u) \in M \times C^m$$

$$\overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z g(z)}\, u - \nabla_{\bar z} g(z)\, \bar u \in T^*$$

$$u \in S^*,$$

that is, the dual of problem (P).

THEOREM 1. (Weak duality theorem). *Let M be a nonempty open convex set in C^n. Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f : M \to C$ be differentiable on M having the real part convex. Let $g : M \to C^m$ be differentiable on M and concave with respect to S. Then*

$$\inf \text{ in (P)} \geqslant \sup \text{ in (S)}.$$

The *proof* is given in [4].

4. Formulation of the dual of the dual. Let M be a nonempty open set in C^n. Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f : M \to C$ and $g : M \to C^m$ both be differentiable on M. It follows that the functions $F : M \times C^m \to C$ and $G : M \times C^m \to C^n$ defined by

(7)
$$F(z, u) = -f(z) + \langle g(z), u\rangle + \langle z, \overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z g(z)} u - \nabla_{\bar z} g(z)\, \bar u\rangle, \quad \text{for all } (z, u) \in M \times C^m,$$

(8)
$$G(z, u) = \overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z g(z)}\, u - \nabla_{\bar z} g(z) \bar u, \text{ for all } (z, u) \in M \times C^m,$$

are differentiable on $M \times C^m$. Then, the problem (D) can be written as

(9)
$$-\inf \operatorname{Re} F(z, u) \text{ subject to } (z, u) \in M \times C^m,\ G(z, u) \in T^*, u \in S^*,$$

and, hence, the dual of (D), that is, the dual of the dual of (P), may be formed. Following the process which yielded the dual (6) of (P), we obtain the dual of (9) in the form

$$\inf \operatorname{Re}[-F(z, u) + \langle G(z, u), t\rangle + \langle u, s\rangle]$$

subject to

$$(z, u, t, s) \in M \times C^m \times C^n \times C^m$$

(10)
$$\overline{\nabla_{(z,u)} F(z, u)} + \nabla_{\overline{(z,u)}} F(z, u) - \overline{\nabla_{(z,u)} (G(z, u), u)} \begin{pmatrix} t \\ s \end{pmatrix} - \nabla_{\overline{(z,u)}} (G(z, u), u) \overline{\begin{pmatrix} t \\ s \end{pmatrix}} = 0$$

$$t \in T, \quad s \in S,$$

because $(T^* \times S^*)^* = T^{**} \times S^{**} = T \times S$.

If $g = (g_1, \ldots, g_m)$ and $u = (u_1, \ldots, u_m)$, then problem (10) can be written in the form

$$\inf \operatorname{Re}[f(z) - \langle g(z), u\rangle - \langle \overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z g(z) u} - \nabla_{\bar z} g(z) \bar u, z - t\rangle + \langle s, u\rangle]$$

subject to

$$(z, u, t, s) \in M \times C^m \times C^n \times C^m$$

(11)
$$[\overline{\nabla^2_{zz} f(z)} + \nabla^2_{\bar z \bar z} f(z) - u^T\, \overline{\nabla^2_{zz} g(z)} - u^H\, \nabla^2_{\bar z \bar z} g(z)]\,(\bar z - \bar t) + [\overline{\nabla^2_{z \bar z} f(z)} + \nabla^2_{\bar z z} f(z) -$$

$$- u^T \overline{\nabla^2_{z\bar z}\, g(z)} - u^H \nabla^2_{\bar z z} g(z)] (z - t) = 0$$

$$g(z) - [\nabla_z g(z)]^T (z - t) - [\nabla_{\bar z} g(z)^T] (\bar z - \bar t) - s = 0$$

$$s \in S, \quad t \in T,$$

where

$$u^T \nabla^2_{zz}\, g(z) = \sum_{k=1}^{m} u_k\, \overline{\nabla_{zz} g_k(z)},\ u^T \overline{\nabla^2_{z\bar z} g(z)} = \sum_{k=1}^{m} u_k \overline{\nabla^2_{z\bar z} g_k(z)},$$

$$u^H \nabla^2_{\bar z z}\, g(z) = \sum_{k=1}^{m} \bar u_k \nabla^2_{\bar z z} g_k(z),\ u^H \nabla^2_{\bar z\bar z} g(z) = \sum_{k=1}^{m} \bar u_k \nabla^2_{\bar z\bar z} g_k(z),$$

for all $(z, u) \in M \times C^m$.

Solving the second set of equalities in (11) for s and substituting this result for s throughout (11) yields the form:

$$\inf \operatorname{Re} [f(z) - \langle \overline{\nabla_z f(z)} + \nabla_{\bar z} f(z),\ z - t\rangle]$$

subject to

$$(z, u, t) \in M \times C^m \times C^n$$

(D²)

$$[\overline{\nabla^2_{zz} f(z)} + \nabla^2_{\bar z\bar z} f(z) - u^T \overline{\nabla^2_{zz} g(z)} -$$

$$- u^H \nabla^2_{\bar z\bar z} g(z)] (\bar z - \bar t) + [\overline{\nabla^2_{z\bar z} f(z)} + \nabla^2_{\bar z z} f(z) -$$

$$- u^T \overline{\nabla^2_{z\bar z} g(z)} - u^H \nabla^2_{\bar z z} g(z)] (z - t) = 0$$

$$g(z) - [\nabla_z g(z)]^T (z - t) - [\nabla_{\bar z} g(z)]^T (\bar z - \bar t) \in S$$

$$t \in T.$$

Remark 1. If problem (P) is a quadratic programming problem in complex space of the following form

$$\inf \operatorname{Re}\ f(z) = (1/2) z^H D z + \langle d, z\rangle$$

subject to

$$g(z) = Az + B\bar z + c \in S$$

$$z \in T,$$

where $D \in C^{n\times n}$ is hermitian; $d \in C^n$; $A, B \in C^{m\times n}$; $c \in C^m$ and $S \subseteq C^m$ and $T \subseteq C^n$ both are closed convex cones, then the dual of the dual, i.e. (D²), is

$$\inf \operatorname{Re} [(1/2) z^H D z + (1/2)(t - z)^H (D^T + \bar D) z + \langle z, d\rangle]$$

subject to

$$(D^T + \bar D)(z - t) = 0$$

$$At + B\bar t + c \in S$$

$$t \in T.$$

If the matrix $(D^T + \bar D)$ is nonsingular, then the dual of the dual is equivalent to primal problem.

Remark 2. If for any $u \in C^m$, the system

$$[\overline{\nabla^2_{zz} f(z)} + \nabla^2_{\bar z\bar z} f(z) - u^T \overline{\nabla^2_{zz} g(z)} - u^H \nabla^2_{\bar z\bar z} g(z)] (\bar z - \bar t) +$$

$$+ [\overline{\nabla^2_{z\bar z} f(z)} + \nabla^2_{\bar z z} f(z) - u^T \overline{\nabla^2_{z\bar z} g(z)} - u^H \nabla^2_{\bar z z} g(z)] (z - t) = 0$$

$$(z, t) \in M \times T,$$

has only solutions with $z = t$, then the dual of the dual, i.e. (D²), is equivalent to primal problem, i.e. (P).

5. Properties. Let X be the set of feasible solutions of primal problem (P), let Y be the set of feasible solutions of dual problem (D) and let YD be the set of feasible solutions of problem (D²). Let $\Phi : M \times C^{m+n} \to C$ be the objective function of problem (D²), i.e.

$$\Phi(z, u, t) = f(z) - \langle \overline{\nabla_z f(z)} + \nabla_{\bar z} f(z),\ z - t\rangle \text{ for all } (z, u, t) \in M \times C^{m+n}.$$

The following theorem is analogous to theorem 1.

THEOREM 2. *Let $M \subseteq C^n$ be a nonempty open convex set. Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ and $g: M \to C^m$ both be twice differentiable on M. If the function $F: M \times C^m \to C$ defined by (7) has convex real part and the function $G: M \times C^n$ defined by (8) is concave with respect to T^*, then*

(12) $$\sup \text{ in (D)} \leqslant \inf \text{ in (D}^2).$$

A result relating the infima in (P) and (D²) is given in the following theorem.

THEOREM 3. *Let $M \subseteq C^n$ be a nonempty open convex set. Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ and $g: M \to C^m$ both be twice diferentiable on M. Then*

(13) $$\inf \text{ in (D}^2) \leqslant \inf \text{ in (P)}.$$

Proof. We distinguish four cases:

(*i*) The primal problem (P) has the feasible set $X = \emptyset$. Then inf in (P) $= +\infty$ and hence the statement of the theorem is proved.

(*ii*) The primal problem (P) is consistent and has an optimal solution z^0. Assume, by contradiction, that

(14) $$\inf \text{ in (D}^2) > \inf \text{ in (P)}.$$

Since $z^0 \in X$, it follows that $(z, u, t) = (z^0, 0, z^0)$ is a feasible solution for (D²). On the other hand, $\operatorname{Re}\ \Phi(z^0, 0, z^0) = \operatorname{Re}\ f(z^0)$ which contradicts (14).

(*iii*) The primal problem (P) is consistent and inf in (P) $= -\infty$. Let $(z^k)_{k\in N}$ be a sequence of feasible solution of problem (P), such that

$$\lim_{k\to+\infty} \mathrm{Re}\ f(z^k) = -\infty.$$

Since for each $k \in N$, we have $z^k \in M$, $g(z^k) \in S$. $0 \in T$, it follows that the point $(z, u, t) = (z^k, 0, z^k)$ is a feasible solution for problem (D^2). Moreover

$$\lim_{k\to+\infty} \mathrm{Re}\ \Phi(z^k, 0, z^k) = \lim_{k\to+\infty} \mathrm{Re}\ f(z^k) = -\infty.$$

Now, if we assume that (14) holds, a contradiction arises. Hence inequality (13) holds.

(*iv*) The primal problem (P) is consistent but has no optimal solution. Let $(z^k)_{k\in N}$ be a sequence of feasible solution of problem (P) such that

$$\lim_{k\to+\infty} \mathrm{Re}\, f(z^k) = \inf \ \text{in (P)} > -\infty.$$

Since for each $k \in N$, we have $z^k \in M$, $g(z^k) \in S$, $0 \in T$, it follows that the point $(z, u, t) = (z^k, 0, z^k)$ is a feasible solution for problem (D^2). Moreover

$$\lim_{k\to+\infty} \mathrm{Re}\ \Phi(z^k, 0, z^k) = \lim_{k\to+\infty} \mathrm{Re}\ f(z^k)$$

Now, if we assume that (14) holds, then a contradiction arises, hence inequality (13) holds.

COROLLARY 1. *Let $M \subseteq C^n$ be a nonempty open convex set. Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f : M \to C$ be twice differentiable function on M, with convex real part and $g : M \to C^m$ be twice differentiable function on M, concave with respect to S such that $F : M \times C^m \to C$ defined by (7) has convex real part and $G : M \times C^m \to C^n$ defined by (8) is concave with respect to T^*. Then*

$$\sup \ \text{in (D)} \leqslant \inf \ \text{in } (D^2) \leqslant \inf \text{in (P)}.$$

Proof. Apply theorems 2 and 3.

THEOREM 4. *Let $M \subseteq C^n$ be an nonempty open set and let $S \subseteq C^m$, $T \subseteq C^n$ both be closed convex cones. Let $f : M \to C$ and $g : M \to C^m$ both be twice differentiable on M. If (z^0, u^0, z^0) is an optimal solution of problem (D^2), z^0 is a feasible solution of primal problem (P) and*

$$\mathrm{Re}\ \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar z} f(z^0)\, z^0, -t^0 \rangle \leqslant 0, \tag{15}$$

then z^0 is an optimal solution of primal problem (P) and

$$\mathrm{Re}\, f(z^0) = \mathrm{Re}\ \Phi(z^0, u^0, t^0). \tag{16}$$

Proof. In view of theorem 3 and from (15) we have inf in (P) $\leqslant \mathrm{Re}\, f(z^0) \leqslant \mathrm{Re}[f(z^0) - \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar z} f(z^0), z^0 - t^0\rangle] = \mathrm{Re}\ \Phi(z^0, u^0, t^0) =$ $=$ inf in $(D^2) \leqslant$ inf in (P), hence z^0 is an optimal solution of problem (P) and (16) holds.

THEOREM 5. *Let M be an open nonempty set and let $S \subseteq C^m$, $T \subseteq C^n$ both be closed convex cones. Let $f : M \to C$ and $g : M \to C^m$ be twice differentiable on M. If (z^0, u^0, t^0) is an optimal solution of problem (D^2), $t^0 \in M$, $\mathrm{Re}[f(z^0) - f(t^0)] \geqslant \mathrm{Re}\ \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar z} f(z^0), z^0 - t^0\rangle$, then t^0 is an optimal solution of primal problem (P) and $\mathrm{Re}\, f(t^0) = \mathrm{Re}\ \Phi(z^0, u^0, t^0)$.*

Proof. In the conditions of theorem we have inf in $(D^2) =$ $= \mathrm{Re}\ \Phi(z^0, u^0, t^0) = \mathrm{Re}[f(z^0) - \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar z} f(z^0), z^0 - t^0\rangle] \geqslant \mathrm{Re}\, f(t^0) \geqslant$ $\geqslant$ inf in (P), which, in view of theorem 3, shows that t^0 is an optimal solution of primal problem (P) and $\mathrm{Re}\ f(t^0) = \mathrm{Re}\ \Phi(z^0, u^0, t^0)$.

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