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THE DUAL OF THE DUAL IN MATHEMATICAL PROGRAMMING IN COMPLEX SPACE

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Abstract. The form and properties of the dual of the dual of a nonlinear programming problem in complex space are investigated.

1. Introduction. Let $A, B \in C^{m \times n}$; $c \in C^m$; $d \in C^n$. Let $S \subseteq C^m$ and $T \subseteq C^m$ both be closed convex cones.

The dual of the linear programming problem in complex space

- min Re $\langle z, d \rangle$ subject to $Az + B\overline{z} + c \in S$, $z \in T$, is given by
- (2) max Re $\langle u, -e \rangle$ subject to $-A^{u}u B^{v}\bar{u} + d \in T^{*}, u \in S^{*}.$ Problem (2) can be written as

-min Re $\langle u, c \rangle$ subject to $(-A^H)u + (-B^T)\overline{u} + d \in T^*, u \in S^*$. Then the dual of (3), that is, the dual of the dual of (1), may be formed. The dual of the dual of (1) is exactly problem (1). Hence, the dual of the dual of a linear programming problem in complex space coincides with the primal problem. If the primal problem is not linear, does the dual of the dual coincide with the primal problem?

In this paper the form and properties of the dual of the dual of a nonlinear programming problem in the complex space are investigated. In the real space the form, properties and computational possibilities of the dual of the dual of a nonlinear programming problem are investigated by Unger and Hurter in [6].

g = M of there exists a neighborhood V of a such that f is differentiable 2. Notation and Preliminaries. Let $C^n(\mathbb{R}^n)$ denote the *n*-dimensional complex (real) space and $C^{m\times n}(R^{m\times n})$ the set of $m\times n$ complex (real) matrices. If A is a matrix or a vector, then A^T , \overline{A} , A^H denote its transpose, complex conjugate and conjugate transpose respectively. For $z=(z_i)\in$ $\in C^n$, Re $z=(\operatorname{Re} z_j)\in R^n$ denotes the real part of z. For $z,v\in C^n:\langle z,v\rangle$ $=v^hz$ denotes the inner product of z and v.

A nonempty set S in C^n is a convex cone if $S+S\subseteq S$ and if $r\in S$

 $\in R, \ r > 0, \ ext{then} \ rS \subseteq S.$

For any nonempty set S in C^n let $S^* = \{v \in C^n : \operatorname{Re}\langle z, v \rangle \ge 0$ for all $z \in S\}$ the polar of S. If S is a nonempty set in C^n , then S^* is a closed convex cone. A nonempty set S in C^n is a closed convex cone if and

Let M be an open set in C^* and let $z^0 \in M$. (i) The function $f: M \to \mathbb{R}$ \rightarrow C is said to be differentiable at z^0 if there exist 2n complex numbers $A_1(z^0), \ldots, A_n(z^0), B_1(z^0), \ldots, B_n(z^0)$ and a function $h(\cdot; z^0): M \to C$ continuous at 20 and vanishing at this point such that

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$$f(z) - f(z^0) = \sum_{j=1}^n A_j(z^0)(z_j - z_j^0) + \sum_{j=1}^n B_j(z^0)(\bar{z}_j - \bar{z}_j^0) +$$
 $+ \|z - z^0\| h(z; z^0) ext{ for all } z \in M.$

If for $z = x + iy \in M(x, y \in R^n)$ we have f(z) = u(x, y) + iv(x, y) then the function f is differentiable at $z^0 = x^0 + iy^0 \in M$ if and only if the functions u and v are differentiable at $(x^0, y^0) \in \mathbb{R}^{2n}$ If we consider the formal differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \text{ and } \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)^{-1}$$

for all
$$j \in \{1, \dots, n\}$$
, we obtain that
$$A_j(z^0) = \frac{\partial f}{\partial z_j}(z^0), \quad B_j(z^0) = \frac{\partial f}{\partial \tilde{z}_j}(z^0) \text{ for all } j \in \{1, \dots, n\}.$$

If $f: M \to C$ is differentiable at $z^0 \in M$, then

The first
$$f: M \to C$$
 is differentiable as $z \in \mathbb{Z}$, where $z \in \mathbb{Z}$ is differentiable as $z \in \mathbb{Z}$, where $z \in \mathbb{Z}$ is differentiable as $z \in \mathbb{Z}$, where $z \in \mathbb{Z}$ is differentiable as $z \in \mathbb{Z}$.

$$riangledown \int f(z^0) = \left(rac{\partial f}{\partial ar{z}_1}(z^0), \, \ldots, rac{\partial f}{\partial ar{z}_n}(z^0)
ight)^T \in C^n.$$

mostlinear progressming problem in the complex space are investigated. The function $f: M \to C$ is said to be differentiable on M if it is differentiated able at any $z^0\in M$; gubinous good manufactures to labor will to lead out to

(ii) The function $f: M \to C$ is said to be twice differentiable at $z \in M$ if there exists a neighborhood V of z such that f is differentiable on V and the functions

on
$$V$$
 and the functions
$$\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial \overline{z}_1}, \dots, \frac{\partial f}{\partial \overline{z}_n} : V \to C$$

are differentiable at z. If f is twice differentiable at z, then z

are differentiable at z. If f is twice differentiable at z, then
$$\nabla^2_{zz} f(z) = \left(\frac{\partial}{\partial z_j} \left(\frac{\partial f}{\partial z_k}(z)\right)\right) \in C^{n \times n},$$

$$\frac{2}{i\bar{z}} f(z) = \left(\frac{\partial}{\partial \bar{z}_j} \left(\frac{\partial f}{\partial \bar{z}_j}(z)\right)\right) \in C^{n \times n},$$

$$abla^2_{ar{z}} \ f(z) = \left(rac{\partial}{\partial ar{z}_j} \left(rac{\partial f}{\partial z_k} \ (z)
ight)
ight) \in C^{n imes n},$$

$$\nabla^{2}_{\bar{z}z} f(z) = \left(\frac{\partial}{\partial z_{j}} \left(\frac{\partial f}{\partial \bar{z}_{k}} (z) \right) \right) \in C^{n \times n}. \tag{1}$$

The function $f: M \to C$ is said to be twice differentiable on M if it is twice differentiable at any $z \in M$.

(iii) The function $g = (g_k): M \to C^m$ is said to be differentiable at $z \in M$ (on M) if for any $k \in \{1, \ldots, m\}$ the function g_k is differentiable at z (on M). If $g = (g_k): M \to C^m$ is differentiable at $z \in M$, then

$$egin{aligned} egin{aligned} egin{aligned\\ egin{aligned} egi$$

If
$$g: M \to C^m$$
 is differentiable at $z \in M$, then $igtriangledown_z ar{g}(z) = igtriangledown_{ar{z}} g(z), \qquad igtriangledown_{ar{z}} g(z) = igtriangledown_{ar{z}} g(z).$

(iv) The function $g = (g_k) : M \to \mathbb{C}^m$ is said to be twice differentiable at $z \in M(\text{on } M)$ if for any $k \in \{1, \ldots, m\}$ the function g_k is twice differentiable at z (on M).

(v) The function $g:M\to C^m$ is said to be convex with respect to closed convex cone $S\subseteq C^m$ if

(4)
$$\lambda g(z) + (\mathbb{I} - \lambda) g(v) - g[\lambda z + (\mathbb{I} - \lambda)v] \in S,$$

for all $\lambda \in [0,1]$ and $z, v \in M$.

If in addition g is differentiable on M, then from (4) it follows that

$$g(z) = g(v) = [\nabla_z g(v)]^T (z-v) = [\nabla_{\bar{z}} g(v)]^T (\bar{z}-\bar{v}) \in S$$

for all $z, v \in M$.

(vi) The function $g: M \to C^m$ is concave with respect to closed convex cone S if g is convex with respect to -S.

(vii) The function $f: M \to C$ is said to be with convex real part if it is convex with respect to the cone $\{v \in C : \text{Re } v \geq 0\}$.

THEOREM L. (West duality theorem). Let M be Winnemply open 3. Background. Let M be a conempty open set in C^u . Let $S\subseteq C^m$ and $\mathcal{I} \subseteq C^n$ both be closed convex cores. Let $f: M \to C$ and $g: M \to C$ $\rightarrow C^m$ both be differentiable functions on M.

We shall consider the following primal problem:

(P) inf Re
$$f(z)$$
 subject to $z \in M$, $g(z) \in S$, $z \in T$.

The dual [1, 4 | of this problem is given by $\sup \operatorname{Re}[f(z) - \langle (g(z), z), (u, v) \rangle]$ subject to making the same state of the same sta

(5)

$$\overline{igtriangledown_{z}f(z)} + igtriangledown_{ar{z}}f(z) - \overline{igtriangledown_{z}(g(z),z)}igg(m{u}{v}igg) - igtriangledown_{ar{z}}(g(z),z)igg(m{u}{v}igg) = 0$$

Since $(S \times T)^* = S^* \times T^*$ and for any $z \in M$, $u \in C^m$, $v \in C^n$ we have

$$igtriangledown_z(g(z),\;z)=(igtriangledown_zg(z)\;\;I)\in C^{n imes(m+n)},$$

$$abla_{ar{z}}(g(z),\ z)=(igtriangledown_{ar{z}}g(z)\ 0)\in C^{n imes(n)},$$

$$\langle (g(z), z), (u, v) \rangle = \langle g(z), u \rangle + \langle z, v \rangle,$$

problem (5) can be written in the form

sup
$$\operatorname{Re}[f(z) - \langle g(z), u \rangle - \langle z, v \rangle]$$

(6)

$$\overline{igtriangledown}_{z}f(z)+igtriangledown_{ar{z}}f(z)-\overline{igtriangledown}_{z}g(z)\,u-igtriangledown_{ar{z}}g(z)\,ar{u}-v=0$$
 $u\in S^*,\ v\in T^*.$

We solve the equality constraints for v, substitute directly for v, and obtain the problem Man a strumur zazirini balala

$$\sup \operatorname{Re} \left[f(z) - \langle g(z), u \rangle - \langle z, \nabla_z \overline{f(z)} + \cdots \right]$$

$$+ igtriangledown_{ar{z}} f(z) - \overline{igtriangledown_{ar{z}} g(z) u} - igtriangledown_{ar{z}} g(z) u
angle
brace \ ext{subject to}$$

$$egin{align} (z,u)\in M imes C^m \ & \overline{igtarting_z f(z)} + igtarting_{ar{z}} f(z) - \overline{igtarting_z g(z)}\, u - igtarting_{ar{z}} g(z)\, ar{u} \in T^* \ & u \in S^* \; , \end{aligned}$$

that is, the dual of problem (P).

THEOREM 1. (Weak duality theorem). Let M be a nonempty open convex set in C^n . Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ be differentiable on M having the real part convex. Let $g: M \to C$ $ightarrow C^m$ be differentiable on M and concave with respect to S. Then

inf in
$$(P) \geqslant \sup in (S)$$
.

The proof is given in [4].

4. Formulation of the dual of the dual. Let M be a nonempty open set in C^n . Let $S \subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let f: $M \to C$ and $g: M \to C^m$ both be differentiable on M. It follows that the functions $F: M \times C^m \to C$ and $G: M \times C^m \to C^n$ defined by

$$F(z, u) = -f(z) + \langle g(z), u \rangle +$$

$$+ \langle z, \overline{\nabla_z f}(z) + \nabla_{\bar{z}} f(z) - \overline{\nabla_z g}(z) u - \overline{\nabla_z g}(z) \bar{u} \rangle, \text{ for all } (z, u) \in M \times C^m,$$

(8)
$$G(z, u) = \overline{\nabla_z f(z)} + \nabla_{\bar{z}} f(z) - \overline{\nabla_z g(z)} u - \nabla_{\bar{z}} g(z) \bar{u}, \text{ for } z = 0$$

$$\text{all } (z, u) \in M \times C^m,$$

are differentiable on $M \times C^m$. Then, the problem (D) can be written as

(9) — inf Re
$$F(z, u)$$
 subject to $(z, u) \in M \times C^m$, $G(z, u) \in T^*$, $u \in S^*$,

and, hence, the dual of (D), that is, the dual of the dual of (P), may be formed. Following the process which yielded the dual (6) of (P), we obtain the dual of (9) in the form

$$\inf_{\substack{\text{inf } \text{Re}[-F(z,u)+\langle G(z,u),t\rangle+\langle u,s\rangle]\\ \text{subject to}\\ (z,u,t,s)\in M\rtimes C^m\times C^n\times C^m}}$$

$$(10) \qquad \overline{\bigtriangledown_{(z,u)}F(z,u)} + \overline{\bigtriangledown_{(\overline{z},u)}F(z,u)} - \overline{\bigtriangledown_{(z,u)}(G(z,u),u)} \binom{t}{s} - \\ - \overline{\bigtriangledown_{(\overline{z},u)}(G(z,u),u)} \overline{\binom{t}{s}} = 0 \\ t \in T, \ \ s \in S,$$

because $(T^* \times S^*)^* = T^{**} \times S^{**} = T \times S$.

If $g = (g_1, \ldots, g_m)$ and $u = (u_1, \ldots, u_m)$, then problem (10) can be written in the form

inf Re
$$[f(z)-\langle g(z),u\rangle-\langle\overline{\nabla}_zf(z)+\overline{\nabla}_{\bar{z}}f(z)-\overline{\nabla}_{\bar{z}}f(z)]$$

$$egin{array}{ll} - igtriangledown_{ar{z}} g(z) \overline{u} - igtriangledown_{ar{z}} g(z) \overline{u}, z-t
angle + \langle s, \ u
angle
bracket \ & ext{subject to} \ & (z, u, t, s) \in M imes C^m imes C^n imes C^m \ & ext{} \end{array}$$

(11)
$$[\nabla^2_{zz}f(z) + \nabla^2_{zz}f(z) - u^T \nabla^2_{zz}g(z) - u^T \nabla^2_{zz}g(z)] = 1 - i (1)$$

$$-\ u^{\scriptscriptstyle H}\ igtriangledown_{ar zar z}^2 g(z)$$
] $(ar z\ -ar t)\ +\ ar [igtriangledown_{ar zar z}^2 f(z)\ +\ igtriangledown_{ar zz}^2 f(z)\ -$

nego v komonom $-u^T \nabla^2_{z\bar{z}} g(z) = u^H \nabla^2_{\bar{z}z} g(z) [(z, -t) = 0]$ set in C". Let $S \subseteq C''$ and $T \subseteq C'$ both he closed convex cones. Let f: and tail) $s \cdot g(z) = [\nabla_{\bar{z}} g(z)]^T (z = t) = [\nabla_{\bar{z}} g(z)^T] (\bar{z} = \bar{t}) \perp s = 0 \text{ and } = M$ we specified S = S and S = S but S = S another S = S

where

$$u^{T} \overline{\bigtriangledown_{zz}^{2}} g(z) = \sum_{k=1}^{m} u_{k} \overline{\bigtriangledown_{zz}g_{k}(z)}, u^{T} \overline{\bigtriangledown_{z\overline{z}}^{2}g(z)} = \sum_{k=1}^{m} u_{k} \overline{\bigtriangledown_{z\overline{z}}^{2}g_{k}(z)},$$

$$u^{H}igtriangledown_{zz}^{2}g(z)=\sum\limits_{k=1}^{m}ar{u}_{k}igtriangledown_{zz}^{2}g_{k}(z),\ u^{H}igtriangledown_{zz}^{2}g(z)=\sum\limits_{k=1}^{m}ar{u}_{k}igtriangledown_{zz}^{2}g_{k}(z),$$

for all $(z, u) \in M \times C^m$. w(z) = (x, y) + (z) = (x, z)

Solving the second set of equalities in (11) for s and substituting this result for s throughout (11) yields the form:

are differentiable on $M \propto G^{o}$. Then, the problem (D) can be written as inf Re $[f(z) - \langle \overline{\nabla_z f(z)} + \nabla_{\bar{z}} f(z), z - t \rangle]$

-inflicer(a, w) subject to (a, v) and without (8)

and hence, the dual of (D), that is, $C_{i,1}^m \times_i C_{i,1}^m \times_i C_{i,1}^m \times_i C_{i,1}^m$ may be formed, Following the process $(z)^{\frac{1}{2}} = \int_{z_z}^{z_z} \int_{z_z}$

$$-u^{H}\nabla_{\bar{z}\bar{z}}^{2}g(z)](\bar{z}-\bar{t})+[\nabla_{z\bar{z}}^{2}f(z)+\nabla_{\bar{z}\bar{z}}^{2}f(z)]$$

$$-u^{H}\nabla_{z\bar{z}}^{2}g(z)-u^{H}\nabla_{\bar{z}z}^{2}g(z)](z-t)=0$$

 $-u^{T}\overline{\nabla_{z\bar{z}}^{2}g(z)}-u^{T}\nabla_{\bar{z}z}^{2}g(z)](z-t)=0$ $g(z)-[\nabla_{z}g(z)]^{T}(z-t)-[\nabla_{\bar{z}}g(z)]^{T}(\bar{z}-\bar{t})\in S$

 $t\in T$. Remark 1. If problem (P) is a quadratic programming problem in complex space of the following form

inf Re $f(z) = (1/2)z^{H}Dz + \langle d, z \rangle$ $Y = {}^{**}Z \times {}^{**}Y = {}^{*}({}^{*}Z \times {}^{*}Y)$ esusped If $y=(g_{11},\ldots,g_{m})$ and $u=(u_{11},\ldots,u_{m})$ then problem (10) can be

 $g(z) = Az + B\bar{z} + c \in S$ written in the form

 $z \in T$, $\angle \{(\gamma) \lor Y \lor (\gamma), \nabla \} = \emptyset \langle n, (x)_{\mathbb{N}} \rangle - \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x)_{\mathbb{N}} \rangle = \emptyset \langle n, (x)_{\mathbb{N}} \rangle + \emptyset \langle n, (x$

where $D \in C^{n \times n}$ is hermitian; $d \in C^n$; $A, B \in C^{m \times n}$; $c \in C^m$ and $S \subseteq C^m$ and $T \subseteq C^n$ both are closed convex cones, then the dual of the dual, i.e. (D^{2}) , is

inf Re $[(1/2)z^HDz + (1/2)(t-z)^H(D^2 + \overline{D})z + \langle z, d \rangle]$ subject to $(D^T + \overline{D})(z-t) = 0$ subject to ...

(11)

 $\begin{array}{lll} At + B\overline{t} + c \in \mathcal{S} \\ t \in T. & = (z)(z+\overline{z}) + (z)(z+\overline{z}) + (\overline{z}-\overline{z})[(z)(z+\overline{z})] \end{array}$

If the matrix $(D^T + \overline{D})$ is nonsingular, then the dual of the dual is equivalent to primal problem.

Remark 2. If for any
$$u \in C^m$$
, the system
$$\left[\overline{\nabla_{zz}^2 f(z)} + \nabla_{\bar{z}z}^2 f(z) - u^T \, \overline{\nabla_{zz}^2 g(z)} - u^H \nabla_{\bar{z}z}^2 g(z) \right] (\bar{z} - \bar{t}) + \\
+ \left[\overline{\nabla_{z\bar{z}}^2 f(z)} + \nabla_{z\bar{z}}^2 f(z) - u^T \overline{\nabla_{z\bar{z}}^2 g(z)} - u^H \overline{\nabla_{z\bar{z}}^2 g(z)} \right] (z - t) = 0 \\
(z, t) \in M \times T,$$

has only solutions with z = t, then the dual of the dual, i.e. (D^2) , is equivalent to primal problem, i.e. (P).

5. Properties. Let X be the set of feasible solutions of primal problem (P), let Y be the set of feasible solutions of dual problem (D) and let YD be the set of feasible solutions of problem (D²). Let $\Phi: M \times C^{m+n} \to \mathbb{R}$ \rightarrow C be the objective function of problem (D²), i.e.

the point (a w, t) - (a 0) for a your block of minds for problem (DS) $\Phi(z,u,t)=f(z)-\langle \overline{\nabla_z\,f(z)}+\overline{\nabla_z\,f(z)},\ z-t
angle \ \ ext{for all } (z,u,t)\in M imes C^{m+n}$ In Ro $\Phi(z^2, 0, z^2) = (H(n^2) \ln^2 K_1 \delta_2 + \max_{i \in A} (cit i))$

The following theorem is analogous to theorem 1.

THEOREM 2. Let $M \subseteq C^n$ be a nonempty open convex set. Let $S \subseteq$ and $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ and $g: M \to C^m$ both be twice differentiable on M. If the function $F: M \times C^m \mapsto O$ defined by (7) has convex real part and the function $G:M\times C''$ defined by (8) is concave with respect to T*, then sooner arise. If no approved constraints

(12) \times M : "I but down sup in (D) \leq inf in (D²) to so that a distinct of the down to be in (D) \leq inf in (D²) to so that (T) at benefit $D \in$

A result relating the infima in (P) and (D2) is given in the following theorem. sup in (D) \leq int in (D²) \leq int in (P).

THEOREM 3. Let $M \subseteq C^n$ be a nonempty open convex set. Let $S \subseteq$ $\subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ and g: $M \to C^m$ both se twice differentiable on M. Then \mathbb{R}^n is the following M $T \subseteq C^*$ both by chosed convex convex left; $M = \mathcal{U}$ and $g \colon M \to C^*$ both (15) dong to material maintains (D2); & inf in (P). To administration of (1) 21, 20 is a fourthle solution of primal problem (P) and

Proof. We distinguish four cases:

(i) The primal problem (P) has the feasible set $X = \emptyset$. Then inf in (P) = $+\infty$ and hence the statement of the theorem is proved.

(ii) The primal problem (P) is consistent and has an optimal solution 20. Assume, by contradiction, that off off off (16).

inf in $(D^2) > \inf$ in (P).

(4) In this exact on (G1) most thus Euler with its work at Aport. (14)Since $z^0 \in X$, it follows that $(z, u, t) = (z^0, 0, z^0)$ is a feasible solution for (D²). On the other hand, Re \oplus $(z^0,0,z^0) = \text{Re } f(z^0)$ which contradicts (P) and (II) inchis

(iii) The primal problem (P) is consistent and inf in (P) = $-\infty$. Let $(z^k)_{k\in\mathbb{N}}$ be a sequence of feasible solution of problem (P), such that

$$\lim_{k \to +\infty} \mathrm{Re} \ f(z^k) = -\infty.$$

Since for each $k \in N$, we have $z^k \in M$, $g(z^k) \in S$. $0 \in T$, it follows that the point $(z, u, t) = (z^k, 0, z^k)$ is a feasible solution for problem (D^2) . Moreover

$$\lim_{k\to +\infty} \ \operatorname{Re} \ \Phi(z^k,\,0,\,\,z^k) = \lim_{k\to +\infty} \operatorname{Re} \ f(z^k) = -\,\,\infty.$$

Now, if we assume that (14) holds, a contradiction arises. Hence inequality (13) holds.

(iv) The primal problem (P) is consistent but has no optimal solution. Let $(z^k)_{k\in\mathbb{N}}$ be a sequence of feasible solution of problem (\hat{P}) such that

$$\lim_{k\to +\infty} \operatorname{Re} f(z^k) = \inf \quad \text{in } (P) > -\infty.$$

Since for each $k \in N$, we have $z^k \in M$, $g(z^k) \in S$, $0 \in T$, it follows that the point $(z, u, t) = (z^k, 0, z^k)$ is a feasible solution for problem (D^2) . Moreover

$$\lim_{k \to +\infty} \operatorname{Re} \Phi(z^k, 0, z^k) = \lim_{k \to +\infty} \operatorname{Re} f(z^k)$$

Now, if we assume that (14) holds, then a contradiction arises, hence inequality (13) holds.

Corollary 1. Let $M \subseteq C^n$ be a nonempty open convex set. Let $S \subseteq$ $\subseteq C^m$ and $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ be twice differentiable function on M, with convex real part and $g: M \to C^m$ be twice differentiable function on M, concave with respect to S such that $F: M \times C^m \rightarrow$ \rightarrow C defined by (7) has convex real part and $G: M \times C^m \rightarrow C^n$ defined by (8) is concave with respect to T*. Then at smill and and a property

sup in (D)
$$\leq$$
 inf in (D²) \leq inf in (P).

Proof. Apply theorems 2 and 3.

THEOREM 4. Let $M \subseteq C^n$ be an nonempty open set and let $S \subseteq C^m$, $T\subseteq C^n$ both be closed convex cones. Let $f:M\to C$ and $g:M\to C^m$ both be twice differentiable on $M.(If(z^0, u^0, z^0))$ is an optimal solution of problem (D2), 20 is a feasible solution of primal problem (P) and

(15)
$$\operatorname{Re} \left\langle \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0) z^0, -t^0 \right\rangle \leqslant 0,$$

then z⁰ is an optimal solution of primal problem (P) and

(16),
$$\operatorname{Re} f(z^0) = \operatorname{Re} \Phi(z^0, u^0, t^0).$$

Proof. In view of theorem 3 and from (15) we have inf in (P) \leq $\leq \operatorname{Re} f(z^0) \leq \operatorname{Re} [f(z^0) - \overline{\langle \nabla_z f(z^0) + \nabla_{\bar{z}} f(z^0), z^0 - t^0 \rangle}] = \operatorname{Re} \Phi(z^0, u^0, t^0) =$ = inf in (D²) \le inf in (P), hence 20 is an optimal solution of problem (P) and (16) holds.

THEOREM 5. Let M be an open nonempty set and let $S \subseteq C^m$, $T \subseteq C^n$ both be closed convex cones. Let $f: M \to C$ and $g: M \to C^m$ be twice differentiable on M. If (z0, u0, t0) is an optimal solution of problem (D2), $t^0 \in M$, $\operatorname{Re}[f(z^0) - f(t^0)] \ge \operatorname{Re} \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0), z^0 - t^0 \rangle$, then t^0 is an optimal solution of primal problem (P) and Re $f(t^0)$ = Re $\Phi(z^0, u^0, t^0)$.

Proof. In the conditions of theorem we have inf in (D^2) $=\operatorname{Re}\,\Phi(z^0,\ u^0,\ t^0)=\operatorname{Re}\,[f(z^0)-\langle\overline{\bigtriangledown_zf(z^0)}+\bigtriangledown_{\bar{z}}f(z^0),\,z^0-t^0\rangle]\geqslant\operatorname{Re}\,f(t^0)\geqslant$ ≥ inf in (P), which, in view of theorem 3, shows that to is an optimal solution of primal problem (P) and Re $f(t^0) = \text{Re } \Phi(z^0, u^0, t^0)$.

REFERENCES

1. Abrams, R. A., Nonlinear programming in complex space: sufficient conditions and duality, J. Math. Anal. Appl., 38(1972) 3, 619-632

2. Ben-Israel, A., Linear equations and inequalities on finite dimensional, real or complex, vector spaces: a unified theory, J. Math. Anal. Appl., 27(1969) 2, 367-389

3. Duca, D. 1., Necessary optimality criteria in nonlinear programming in complex space with differentiability, L'Analyse numérique et la théorie de l'approximation, 9(1980) 2, 163-179

4. Duca, D. 1., Mathematical programming in complex space, Doctoral thesis, University of Cluj-Napoca, Cluj-Napoca, 1981

5. Duca, D. I., Duality in mathematical programming in complex space. Converse theorems L'Analyse numérique et la théorie de l'approximation, 13 (1984) 1, 15-22

6. Unger, P. S. and Hurter, A. P., Some properties of the dual of the dual in nonlinear programming, Cahiers du C.E.R.O., 14(1972) 3, 169-176

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