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ON THE CONVERGENCE OF A CLASS OF ITERATIVE METHODS IN FRÉCHET SPACES

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In paper [1], a class of iterative methods for solving the operatorial equation

(1) $$P(x) = 0$$

is given, where $P : X \to X$ is a continuous nonlinear operator, X being Fréchet space.

The goal of this paper is the improving of the rapidity of convergence of this method. For this purpose, equation (1) will be written

(1′) $$P(x) \equiv x - F(x) = \theta$$

and we consider the method given by the algorithm

(2) $$x_{n+1} = x_n - \wedge_n P(x_n) - \wedge_n (I + aR_n)^{-1} R_n P(x_n)$$

where $a \in \mathbb{R}$, $\wedge_n = [x_n, u_n; P]^{-1}$ is the inverse of the first order divided difference of operator P in the node x_n, $u_n = F(x_n)$ and

$$R_n = [x_n, u_n, v_n; P]\overline{\wedge}_n P(u_n) \wedge_n$$

having $v_n = F(u_n)$, $[x_n, u_n, v_n; P]$ is the second order divided difference of P in nodes x_n, u_n, v_n, $\overline{\wedge}_n = [u_n, v_n, P]^{-1}$ the inverse of the first order divided difference of operator P in nodes u_n, v_n, I being the identical operator of the space.

In what follows, we shall denote by $)|x|($ the quasinorm of an element x of space X, defined by a distance d invariant to translation. i.e. $)|x|(= d(x, 0)$ [2].

THEOREM. *Suppose that a nod $x_0 \in X$ and the constants $\eta_0 > 0$, $B > 0$, $K \geqslant 1$, $M > 0$ and $N > 0$ exists, so that;*

1. $)|P(x_0)|(\leqslant \eta_0$;
2. *for any $x', x'', x''', x^{IV} \in S(x_0, r)$ the following boundings*

(a) $)|[x', x''; P]^{-1}|(\leqslant B$

(b) $)|x', x''; F]|(\leqslant K$;

(c) $\| [x', x'', x'''; P \| \leqslant M$;

(d) $\| [x', x'', x'''; P)] - [x'', x''', x^{IV}; P] \| \leqslant$
$\leqslant N) \| x' - x^{IV} \|$;

hold;

3\. $E_0 h_0 < 1$ *where*

$$E_0^2 = \{[|1 + a|(1 + h_0) + 1](1 - |a|h_0) + \frac{1}{k}(1 + |1 + a|h_0]\}/(1 - |a|h_0)^2 +$$

$$+ \frac{N[1 + (1 - |a|)h_0^2[h_0 + (2 + k^2)(1 - |a|h_0)}{BM^2K(1 - |a|h_0)(1 - |a|h_0)^2}$$

$$h_0 = B^2 MK \eta_0, \quad |a|h_0 < 1,$$

then, in the ball $S(x_0, r)$ *where* $r = (1 + k)\eta_0 + K^2 L$, *and*

$$L = B\eta_0 + \frac{1 + (1 - |a|)h_0}{1 - |a|h_0} \sum_{n=0}^{\infty} (E_0 h_0)^{3^n - 1},$$

equation (1) has in S *one and only one solution, which is the limit of the sequence* (x_n) *generated by algorithm* (2), *the rapidity of convergence being characterized by the inequality*

(3) $$\| x_n - x^* \| \leqslant (E_0 h_0)^{3^n - 1} L.$$

Proof. We note that $u_n, v_n \in S(x_0, r)$. Indeed, from the hypothesis we get

$$\| x_0 - u_0 \| = \| x_0 - F(x_0) \| = \| P(x_0) \| \leqslant \eta_0 < r$$

$$\| x_0 - v_0 \| = \| x_0 - F(u_0) \| = \| x_0 + F(x_0) - [x_0, u_0; F](x_0 - u_0) \| \leqslant$$
$$\leqslant (K + 1)\eta_0 < r.$$

Based on conditions 1 and 2 we get

$$\| P(u_0) \| = \| u_0 - F(u_0) \| =$$
$$= \| [x_0, u_0]; F](u_0 - x_0) \| \leqslant K \eta_0;$$

$$\| P(v_0) \| = \| v_0 - F(v_0) \| = \| F(u_0) - F_0) \| \leqslant$$
$$\leqslant \| [u_0, v_0; F] \| (.) | u_0 - v_0 | \leqslant k^2 \eta_0.$$

From conditions 1, 2 and 3 it also results

$$\| aR_0 \| \leqslant |a|h_0 < 1$$

so the operator $H = (I + aR_0)^{-1}$ exists and

$$\| H \| \leqslant \frac{1}{1 - |a|h_0}.$$

It results that the approximate x_1 can be computed with the help of algorithm (2) and that

$$\| x_1 - x_0 \| \leqslant \frac{B\eta_0}{1 - |a|h_0}[1 + (1 - |a|)h_0] \leqslant r$$

so $x_1 \in S$.

We show that, for any x, conditions 1 – 3 of the theorem are satisfied. Indeed, taking into account the formula

$$P(x_1) = (1 + a)(I + aR_0)^{-1} R_0 R_0 P(x_0) + [x_0, u_0, v_0; P] \wedge_0$$
$$[(I + aR_0)^{-1} R_0 P(x_0) - [x_0, u_0, v_0; P | (x_0 - v_0) \wedge_{-1} P(u_0)]$$
$$\wedge_0 (I + aR_0)^{-1} [I + (1 + a)R_0] P(x_0) + [x_1, x_0, u_0, v_0; P]$$
$$(x_1 - v_0)(x_1 - u_0)(x_1 - x_0)$$

and the evaluations

$$\| x_1 - x_0 \| \leqslant T[1 + (1 - |a|h_0],$$
$$\| x_1 - u_0 \|) \leqslant T[1 + (1 - |a|)h_0],$$
$$\| x_1 - v_0 | \leqslant T[h_0 + (2 + k^2)(1 - |a|h_0)],$$

where $T = \dfrac{\beta \eta_0}{1 - |a| h_0}$ we get

$$\| P(x_1) \| \leqslant (E_0 h_0)^2 \eta_0 = \eta_1 < \eta_0$$

so condition 1 is satisfied.

Conditions 2 are, obviously, satisfied.

Condition 3 is verified setting $h_1 = BMK \eta_1$ and taking into account that $h_1 < B^2MK\eta_0 = h_0$, and it results $E_1 < E_0$ and $E_1 h_1 < E_0 h_0$.

Using the induction, we show that any approximation $x_n \in S(x_0, r)$ can be built with the help of (2) and we get

(4) $$\| P(x_n) \| \leqslant \eta_n) \qquad \| P(u_n) \| \leqslant K\eta_n \qquad \| P(v_n) \| \leqslant K^2 \eta_n$$

$$\| X_{n+1} - x_n \| \leqslant \frac{B\eta_n}{1 - |a|h_n}[1 + (1 - |a|)h_n]$$

(5) $$\| x_{n+1} - u_n \| \leqslant \frac{kB\eta_n}{1 - |a|h_n}[1 + (1 - |a|)h_n].$$

$$\| x_{n+1} - v_n \| \leqslant \frac{B\eta_n}{1 - |a|h_n}[h_n + (2 + k^2)(1 - |a|h)_n]$$

with $u_n, v_n \in S(x_0, r)$ and

$$\| P(x_{n+1} \| \leqslant (E_n h_n)^2 \eta_n = \eta_{n+1}$$

Because

$$h_n = B^2 MK \eta_n \leqslant \frac{1}{E_0} (E_0 h_{n-1})^3$$

it results

(7) $$h_n \leqslant \frac{1}{E_0} (E_0 h_0)^{3^n}$$

(8) $$\eta_{n+1} \leqslant (E_0 h_0)^{3^{n+1}-1} \eta_0.$$

We get, then

$$\| x_{n+1} - x_n \| \leqslant \frac{B\eta_0[1 + (1 - |a|)h_0}{1 - |a| h_0} (E_0 h_0)^{3^n - 1}$$

and so

(9) $$\| x_{n+p} - x_n \| \leqslant \frac{B\eta_0 [1 + (1 - |a| h_0}{1 - |a| h_0} (E_0 h_0)^{3^n - 1} \sum_{k=1}^{p} (E_0 h_0)^{3^n(3^{k-1}-1)}.$$

The space X being complete, it results that there exists

$$\lim_{n\to\infty} x_n = x^*$$

If in (9) we put $p \to \infty$ we obtain the inequality (3). The fact that $P(x^*) = 0$ results from (6), taking into account (8) and the continuity of P.

To prove the uniqueness of x^*, suppose that there exists a solution $\tilde{x} \neq x^*$ of the equation (1), so $P(\tilde{x}) = \theta$.

We have, then

$$\| x_n - \tilde{x} \| = \| [x_n, \tilde{x} ; P]^{-1} [x_n, \tilde{x} ; P] (x_n - \tilde{x}) \| \leqslant$$
$$\leqslant B \| P(x_n) \| \leqslant B (E_0 h_0)^{3^n - 1} \eta_0,$$

so $\lim x_n = \tilde{x}$ and so $\tilde{x} = x^*$.

Remark. In the case of a Banach space, an analogous theorem has been proved by G. Goldner [3]. In this case, evaluations even for the third order of divided differences of P are demanded.

REFERENCES

1. Groze, S. *A Class of Iterative Methods in Fréchet Spaces.* Research Seminars, Preprint no. 5, 1987, pp. 28–35.
2. Collatz, L. *Funktionalanalysis und Numeriche Matematik*, Springer, Verlag Berlin, 1965,
3. Goldner, G. *Asupra convergenței unei clase de metode iterative în spații Banach*, Anal. St. ale Univ. Iași, Matematica, Tom XV, 1969, fasc. 1, pp. 187–190.

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