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LINEAR INTEGER AND LINEAR RATIONAL INEQUALITIES

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The purpose of this paper is to establish some fundamental theorems for linear integer and linear rational inequalities. In this paper we shall use the following symbols :

N = the set of natural numbers ; $N^* = N \setminus \{0\}$;
Z = the set of integer numbers ; $Z^* = Z \setminus \{0\}$;
Q = the set of rational numbers ; $Q^* = Q \setminus \{0\}$;
Q_+ = the set of nonnegative rational numbers ; $Q_+^* = Q_+ \setminus \{0\}$.

Let $m, n \in N^*$. By $M_{m\times n}(Z)$ we denote the set of m × n matrices which elements are integer numbers and by $M_{m\times n}(Q)$, the set of $m\times n$ matrices which elements are rational numbers.

THEOREM 1. *For any given matrix* $A \in M_{p\times n}(Z)$, *the systems*

(I) $$\begin{cases} Ax \geqslant 0_p \\ x \in Z^n \end{cases}$$

and

(II) $$\begin{cases} A^T y = 0_n \\ y \in N^p \end{cases}$$

possess solutions x^0 *and* y^0 *satisfying* $\langle A_1, x^0\rangle + y_1^0 > 0$, *where* $A_1^T = (a_{11},\ldots, a_{1n})$.

Proof. The proof is by induction on p. For $p = 1$, two cases are possible :

1) $A_1 = 0_n$. Then $x^0 = 0_n$ is a solution of the system (I), $y^0 = 1$ a solution of the system (II) and $\langle A_1, x^0\rangle + y_1^0 = 0 + 1 = 1 > 0$.

2) $A_1 \neq 0_n$. Because $A_1 \neq 0_n$, we have $\langle A_1, A_1\rangle = \sum_{j=1}^{n} a_{1j}^2 > 0$. Since $A_1 \in Z^n$, we get that $x^0 = A_1$ is a solution of the system (I), $y^0 = 0$, a solution of the system (II) and $\langle A_1, x^0\rangle + y_1^0 > 0$.

Therefore, for $p = 1$ the conclusion of the theorem is true.

Now assume that the theorem is true for a matrix A of p rows and proceed to prove it for a matrix

$$\bar{A} = \begin{bmatrix} A \\ A_{p+1} \end{bmatrix} = \begin{bmatrix} A_1 \\ \cdots \\ A_p \\ A_{p+1} \end{bmatrix}$$

of $p+1$ rows, where $A_i = (a_{i1}, \ldots, a_{in})^T \in Z^n$ for all $i \in \{1, \ldots, p+1\}$. By applying the theorem to A, we have $x^0 \in Z^n$, $y^0 \in N^p$ satisfying

(1) $$Ax^0 \geqslant 0_p,$$

(2) $$A^T y^0 = 0_n,$$

(3) $$\langle A_1, x^0\rangle + y_1^0 > 0$$

Two cases are possible: $\langle A_{p+1}, x^0\rangle \geqslant 0$ or $\langle A_{p+1}, x^0\rangle < 0$. If $\langle A_{p+1}, x^0\rangle \geqslant 0$, taking $\bar{y} = (y^0, 0)$ we have $\bar{A}x^0 \geqslant 0_p$, $\bar{y} \in N^{p+1}$, $\bar{A}\bar{y} = 0_n$ and $\langle \bar{A}_1, x^0\rangle + y_1^0 > 0$, which states the conclusion of the theorem for $\bar{A}$.

Let now $\langle A_{p+1}, x^0\rangle < 0$. Then taking

(4) $$t = -\langle A_{p+1}, x^0\rangle, \quad q_j = \langle A_j, x^0\rangle \quad (j = 1, \ldots, p),$$

we have

(5) $$t \in N^* \text{ and } q_j \in N \quad (j = 1, \ldots, p).$$

Let

(6) $$B = \begin{bmatrix} B_1 \\ \cdots \\ B_p \end{bmatrix} = \begin{bmatrix} tA_1 + q_1 A_{p+1} \\ \cdots \\ tA_p + q_p A_{p+1} \end{bmatrix}$$

For all $j \in \{1, \ldots, p\}$ we have

(7) $$\langle B_j, x^0\rangle = \langle tA_j + q_j A_{p+1}, x^0\rangle = tq_j - q_j t = 0.$$

By applying the theorem to B, we get $v = (v_1, \ldots, v_n) \in Z^n$, $u = (u_1, \ldots, u_p)$, $u \in N^p$ satisfying:

(8) $$Bv \geqslant 0_p, \ B^T u = 0_n, \ B_1 v + u_1 > 0.$$

Let

(9) $$\bar{u} = (tu, q_1 u_1 + \ldots + q_p u_p).$$

Because $t \in N^*$, $q_j \in N$ $(j = 1, \ldots, p)$ and $u \in N^p$, we have

(10) $$\bar{u} \in N^{p+1},$$

and

$$\bar{A}^T \bar{u} = tA^T u + A_{p+1}^T \cdot \sum_{j=1}^{p} q_j u_j.$$

Then

(11) $$\bar{A}^T \bar{u} = B^T u = 0_n.$$

Let $w = tv + \langle A_{p+1}, v\rangle \cdot x^0$. Because $t \in N^*$, $v \in Z^n$, $A_{p+1} \in Z^n$ and $x^0 \in Z^n$, it follows that

(12) $$w \in Z^n.$$

We have

(13) $$\langle A_{p+1}, w\rangle = t\langle A_{p+1}, v\rangle + \langle A_{p+1}, v\rangle\langle A_{p+1}, x^0\rangle = \langle A_{p+1}, v\rangle(t - t) = 0.$$

and

$$\langle A_j, w\rangle = t\langle A_j, v\rangle + \langle A_{p+1}, v\rangle\langle A_j, x^0\rangle = t\langle A_j, v\rangle + q_j\langle A_{p+1}, v\rangle.$$

From (5) we get $A_j = \frac{1}{t} B_j - \frac{q_j}{t} A_{p+1} (j = 1, \ldots, n)$.

Hence

(14) $$\langle A_j, w\rangle = t\langle \frac{1}{t} B_j, v\rangle - t\frac{q_j}{t}\langle A_{p+1}, v\rangle + q_j\langle A_{p+1}, v\rangle = \langle B_j, v\rangle$$

for all $j = 1, \ldots, n$

From (14) and (13) it results that

(15) $$\bar{A}w = \begin{bmatrix} Aw \\ \langle A_{p+1}, w\rangle \end{bmatrix} = \begin{bmatrix} Bv \\ 0 \end{bmatrix} \geqslant 0_{p+1}$$

Finally, from (14) and (8) we have

(16) $$\langle \bar{A}_1, w\rangle + u_1 = \langle B_1, v\rangle + u_1 > 0.$$

Relations (10), (11), (12), (15) and (16) state the theorem for $\bar{A}$.

Likewise we can prove:

THEOREM 1'. *For any given matrix $A \in M_{p\times n}(Q)$, the system*

(I') $$\begin{cases} Ax \geqslant 0_p \\ x \in Q^n \end{cases}$$

and

(II)' $$\begin{cases} A^t y = 0_n \\ y \in Q^p \end{cases}$$

possess solutions x^0 and y^0 satisfying $\langle A_1, x^0\rangle + y_1^0 > 0$, where $A_1^T = (a_{11}, \ldots, a_{1n})$.

THEOREM 2. *For any given matrix $A \in M_{p\times n}(Z)$ the systems (I) and (II) possess solutions x^0 and y^0 satisfying $Ax^0 + y^0 > 0_p$.*

Proof. In theorem 1 the row A_1 played a special role. By renumbering the rows of A, any other row, say A_i, can play the same role. Hence, by theorem 1, for all $i \in \{1, \ldots, p\}$ there exist $x^i \in Z^n$ and $y^i \in N^p$ such that

(17) $$Ax^i \geqslant 0_p, \ A^T y^i = 0_n, \ A_i x^i + y_i^i > 0.$$

Consider $x^0 = x^1 + \ldots + x^p$ and $y^0 = y^1 + \ldots + y^p$. Obviously, we have $x^0 \in Z^n$ and $y^0 \in N^p$. From (17) we obtain

$$(18) \quad Ax^0 = Ax^1 + \ldots + Ax^p \geqq 0_p, \quad Ay^0 = Ay^1 + \ldots + Ay^p = 0_n,$$

For all $i \in \{1, \ldots, p\}$ we have $\langle A_i, x^0\rangle + y_i^0 = \langle A_i, x^i\rangle + y_i^i + \sum_{\substack{k=1 \\ k\neq i}}^{p} \langle A_i, x^k\rangle + \sum_{\substack{k=1 \\ k\neq i}}^{p} y_i^k \geqq \langle A_i, x^i\rangle + y_i^i > 0$. Hence

$$(19) \qquad Ax^0 + y^0 > 0_p.$$

As $x^0 \in Z^n$, $y^0 \in N^p$, relations (18) and (19) show that x^0 is a solution of the system (I), y^0 a solution of the system (II), and $Ax^0 + y^0 > 0$.

Likewise we can prove :

THEOREM 2′ *For any given matrix* $A \in M_{p\times n}(Q)$, *the system* (I′) *and* (II′) *possess solution* x^0 *and* y^0 *satisfying* $Ax^0 + y^0 > 0_p$.

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