MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION,
Tome 19, N° 1, 1990, pp. 53–58

PROPERTIES OF BOUNDED CONVEX SEQUENCES

IGOR Ž. MILOVANOVIĆ, MILAN A. KOVAČEVIĆ and GH. TOADER
(Niš) (Niš) (Cluj-Napoca)

Abstract. Some properties of the bounded convex sequences of order $m \geqslant 2$ are considered in this paper. As it is shown, these properties are in direct connection with the convergence of a certain class of series.

1. Introduction. When studying convergence conditions, summability and other properties of series the knowledge of sequence properties is of a decisive importance. Due to their specific nature, different classes of sequences, for example classes of bounded convergent, convex, star-shaped and other sequences (see for example [1] – [16]) are studied hardly. In this paper some properties of bounded convex sequences with order of convexity $m(\geqslant 2)$ and their relation with a certain class of real series is considered.

Let us first introduce some notation and definitions. Denote by S_m the class of real sequences (a_n), $n \in N_0$, with the following properties

(1.1) $$M_2 \leqslant a \leqslant M_1, \quad n = 0, 1, \ldots \; (M_1, M_2 = \text{const.}),$$

and

(1.2) $$\nabla^r a_n \geqslant 0, \quad r = 2, \ldots, m \; (m \geqslant 2); \quad n = 0,1, \ldots$$

where

$$\nabla^r a_n = (-1)^r \Delta^r a_n, \quad (\Delta a_n = a_{n+1} - a_n, \quad \Delta^r a_n = \Delta(\Delta^{r-1} a_n)).$$

Let $(s)_p = s(s+1)\ldots(s+p-1)$ and $V_n^r = \binom{n+r}{r}$ for each $p = 1, 2, \ldots, s = 0, 1, \ldots, n = 0, 1, \ldots$ We shall also quote some results, known in the literature which are in relation to those obtained in this paper.

For sequences belonging to S_2 class, the following result is proved in paper [13]:

THEOREM A. *The sequence (a_n), $n \in N_0$, belongs to S_2 class if and only if there is a sequence (b_n), $n \in N_0$, such that*

$$a_n = \sum_{k=0}^{n} (n - k + 1)\, b_k, \quad n \geqslant 0,$$

$$b_k \geqslant 0 \quad \textit{for } k \geqslant 2,$$

$$n \sum_{k=0}^{n} b_k \to 0, \quad n \to \infty,$$

$$\sum_{k=0}^{\infty} (k+1) b_{k+1} + 1 < +\infty .$$

The following results can be found, for example in the paper [10]:

THEOREM B. *Let the sequence* (a_n), $n \in N_0$, *satisfy the following properties*: $\lim_{n\to\infty} a_n = 0$ *and* $\nabla^2 a_n \geqslant 0$ *for* $n = 0, 1, \ldots$. *Then* $\nabla a_n \geqslant 0$,

$$\lim_{n\to\infty} n \nabla a_n = 0 \quad \textit{and} \ \sum_{n=0}^{\infty} (n+1) \nabla^2 a_n = \sum_{n=0}^{\infty} \nabla a_n = a_0 .$$

THEOREM C. *For any convergent series* $\sum_{n=1}^{\infty} a_n$, *we have* $na_n \to 0$ *in the sense of Cesàro.*

The following result is given in monograph [3]:

THEOREM D. *If a sequence* (a_n), $n \in N_0$, *belongs to* S_2 *class, then*:

(a) *the sequence* (a_n), $n \in N_0$ *is decreasing*;

(b) $\lim_{n\to\infty} n \nabla a_n = 0$;

(c) *the series* $\sum_{n=0}^{\infty} (n+1) \nabla^2 a_n$ *is a convergent one and its sum is* $a_0 - \lim_{n\to\infty} a_n$.

2. Main results. Before presenting the main results of this paper, we shall prove several lemmas giving specific properties of the sequences belonging to S_m class.

LEMMA 1. *If a sequence* (a_n), $n \in N_0$, *belongs to* $S_m (m \geqslant 2)$ *class then the following inequality*

(2.1) $$0 \leqslant \nabla^r a_n \leqslant (M_1 - M_2) \frac{V_n^r}{V_n^{2r}} = (M_1 - M_2) \frac{(2r)!}{r!\,(n+r+1)_r},$$

holds for $r = 1, \ldots, m-1$.

Proof. As $\nabla^r a_n \geqslant 0$, for $r = 2, \ldots, m$ the inequality

(2.2) $$\nabla^{r-1} a_n \geqslant \nabla^{r-1} a_{n+1} \quad (n = 0, 1, \ldots).$$

is valid. According to Theorem *D* we conclude that the following implication

(2.3) $$(a_n) \in S_2 \Rightarrow \nabla a_n \geqslant 0$$

holds.

As (a_n), $n \in N_0$, belongs to S_m class $\nabla^r a_n \geqslant 0$ holds for $r = 2, \ldots, m$, then, taking into account (2.3), we prove the left inequality in (2.1). The right side of the inequality (2.1) shal be proved by means of mathematical induction. For $r = 1$ we have

$$(M_1 - M_2) V_n^1 = (M_1 - M_2)(n+1) =$$

$$= M_1(n+1) - M_2(n+1) \geqslant \sum_{k=0}^{n} a_k - M_2(n+1) =$$

$$= \sum_{k=0}^{n} (k+1) \nabla a_k + (n+1) a_{n+1} - M_2(n+1) \geqslant \nabla a_n \sum_{k=0}^{n} (k+1) = V_n^2 \nabla a_n,$$

i.e.

$$0 \leqslant \nabla a_n \leqslant (M_1 - M_2) \frac{V_n^1}{V_n^2}.$$

Let us suppose that (2.1) is valid for some $r = p \, (1 \leqslant p \leqslant m-2)$, i.e.

(2.4) $$0 \leqslant \nabla^p a_n \leqslant (M_1 - M_2) \frac{V_n^p}{V_n^{2p}}.$$

According to equality $\sum_{i=0}^{k} V_i^S = V_k^{S+1}$ and the inductive assumption (2.4) we have

$$(M_1 - M_2) V_n^{p+1} = (M_1 - M_2) \sum_{k=0}^{n} V_k^p \geqslant \sum_{k=0}^{n} V_k^{2p} \nabla^p a_k =$$

$$= \sum_{k=0}^{n} V_k^{2p+1} \nabla^{p+1} a_k + V_n^{2p+1} \nabla^p a_{n+1} \geqslant \nabla^{p+1} a_n V_n^{2p+2},$$

i.e.,

$$\nabla^{p+1} a_n \geqslant (M_1 - M_2) \frac{V_n^{p+1}}{V_n^{2p+2}},$$

which had to be proved.

Using Abel's lemma we directly obtain the following result:

LEMMA 2. *For each sequence of real numbers* (a_n), $n \in N_0$, *the equality*

(2.5) $$a_n = a_0 - \sum_{k=1}^{n} V_{n-k}^k \nabla^k a_{n-k} - \sum_{k=0}^{n-p-1} V_k^p \nabla^{p+1} a_k, \quad \left(\sum_{k=1}^{0} = 0 \right),$$

holds, where $p < n$.

LEMMA 3. *Let the sequence (a_n), $n \in N_0$, belong to the class $S_m (m \geqslant 2)$. Then, equalities*

$$\lim_{n\to\infty} V^k_{n-k} \nabla^k a_{n-k} = 0 , \tag{2.6}$$

for $k = 1, \ldots, m-1$, *and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} V_k^{m-1} \nabla^m a_k = a_0 - \lim_{n\to\infty} a_n \tag{2.7}$$

hold.

Proof. For $m = 2$ Lemma 3 is proved in Theorem A (i.e. Theorem D). Assume that Lemma 3 holds for $k = 1, \ldots, m-2$. According to Lemma 2 we have :

$$a_n = a_0 - \sum_{k=1}^{m-2} V^k_{n-k} \nabla^k a_{n-k} - \sum_{k=0}^{n-m+1} V_k^{m-2} \nabla^{m-1} a_k ,$$

and by the inductive assumption

$$\lim_{n\to\infty} V^k_{n-k} \nabla^k a_{n-k} = 0$$

for $k = 1, \ldots, m-2$. On this basis, we conclude that the series $\sum_{k=0}^{\infty} V_k^{m-2} \nabla^{m-1} a_k$ is a convergent one. According to Theorem C we obtain a sequence $(n V_n^{m-2} \nabla^{m-1} a_n)$, i.e. $(V_n^{m-1} \nabla^{m-1} a_n)$ which tends to zero in Cesàro sense. In other words the equality

$$\lim_{n\to\infty} \frac{V_0^{m-1} \nabla^{m-1} a_0 + \ldots + V_n^{m-1} \nabla^{m-1} a_n}{n+1} = 0, \tag{2.8}$$

holds. Let us prove that the sequence $(V_n^{m-1} \nabla^{m-1} a_n)$ tends toward zero. Assume that it is not true. Then, there would be a constant $C (\geqslant 0)$ such that beginning from some index n the inequality $V_n^{m-1} \nabla^{m-1} a_n \geqslant C$ holds. According to (2.2) for each $k \leqslant n$ the following inequality

$$\Delta^{m-1} a_k \geqslant \nabla^{m-1} a_n \geqslant \frac{C}{V_n^{m-1}}$$

holds. On the other hand, the relation

$$\frac{V_0^{m-1} \nabla^{m-1} a_0 + \ldots + V_n^{m-1} \nabla^{m-1} a_n}{n+1} \geqslant C \frac{V_n^m}{(n+1)\, V_n^{m-1}} \xrightarrow[n\to\infty]{} \frac{C}{m} \neq 0,$$

is in opposition with equality (2.8). It contradicts the assumption that the sequence $(V_n^{m-1} \nabla^{m-1} a_n)$ does not tend to zero. It also means that (2.6) holds even for $k = m-1$. If we substitute $p = m-1$ in (2.5), the assumption (2.7) is directly obtained from the equality

$$\sum_{k=0}^{n-m} V_k^{m-1} \nabla^m a_k = a_0 - \sum_{k=1}^{m-1} V^k_{n-k} \nabla^k a_{n-k} - a_n .$$

From paper [14] we directly obtain the following result.

LEMMA 4. *The sequence of real numbers (a_n), $n \in N_0$, has the property $\nabla^r a_n \geqslant 0$, if and only if there is a sequence (b_n), $n \in N_0$, so that $b_n \geqslant 0$ for $n \geqslant r$ for which equality*

$$a_n = (-1)^r \sum_{k=0}^{n} V^{r-1}_{n-k}\, b_k \tag{2.9}$$

holds.

According to the given lemmas we immediately obtain the following result :

THEOREM 1. *The sequence (a_n), $n \in N_0$, belongs to S_m class if and only if there is a sequence (b_n), $n \in N_0$, such that :*

$$a_n = (-1)^m \sum_{k=0}^{n} V^{m-1}_{n-k}\, b_k \tag{2.10}$$

$$(-1)^{m+j} \sum_{k=0}^{n} V^{m-j-1}_{n-k}\, b_k \geqslant 0 \quad \text{for } j = 2, \ldots, m-1 ;\ n \geqslant j \tag{2.11}$$

$$b_k \geqslant 0 \quad \textit{for } k \geqslant m , \tag{2.12}$$

$$V^k_{n-k} \sum_{i=0}^{n} V^{n-k-1}_{n-i}\, b_i \xrightarrow[n\to\infty]{} 0, \quad \textit{for } k = 1, \ldots, m-1, \tag{2.13}$$

and

$$\sum_{k=0}^{\infty} V_k^{m-1}\, b_{k+m} \ \textit{is convergent.} \tag{2.14}$$

COROLLARY 1. *The sequence of real numbers (a_n), $n \in N_0$, has the following properties $\nabla^r a_n \geqslant 0$, for $r = 2, \ldots, m$ and $\lim_{n\to\infty} a_n = 0$, if and only if there is a sequence (b_n), $n \in N_0$, such that properties (2.10) — (2.13) and $\sum_{k=0}^{\infty} V_k^{m-1}\, b_{k+m} = a_0$ are satisfied.*

REFERENCES

1. Bary, N. K., *A treatise on trigonometric series*, New York, Pergamon Press, 1964.
2. Bray, W., Stanojević, Č. V., *Tauberian L^1 — convergence classes of Fourier series*, Trans. Amer. Math. Soc., **275**(1983), 59—69.
3. Edwards, R. E., *Fourier series*, Vol. 1 Springer Verlag, 1979.
4. Fomin, G. A., *On some conditions for the convergence classes of Fourier series in L metric*, Mat. Zametki **21**(1977), 587—592. (Russian).
5. Fomin, G. A., *On a class of trigonometric series*. Mat. Zametki **23**(1978), 213—222. (Russian).
6. Fomin, G. A., *On the convergence of Fourier series in the mean*, Mat. Sbornik **110** (1979), 215—265. (Russian).
7. Garrett. J. W., Rees, C. S. Stanojević Č. V., *L^1 convergence of Fourier series with coefficients of bounded variation*, Proc. Amer. Math. Soc. **60**(1976), 68—71.

8. Hardy, G. H., Rogosinski, W. W., *Fourier series*, New York, Cambridge Univ. Press, 1956.
9. Kolmogorov, A. N., *Sur l'ordre de grandeur des coefficients de la série de Fourier—Lebesgue*, Bull. Internat. Acad. Polon. Sci. Lettres Cl. Sci. Math. Natur. Ser. A (1923), 83—86.
10. Reade, J. B., *On the order of magnitude of Fourier coefficients*, SIAM J. Math Anal. **17**(1986), 469—476.
11. Stanojević, Č. V., *Classes of L^1-convergence of Fourier and Fourier—Stieltjes series* Proc. Amer. Math. Soc. **82**((1981), 209—215.
12. Tanović—Miller, N., *On a paper of Bojanić and Stanojević*, Rend. Circ. Mat. Palermo, Ser II, **34**(2) (1985), 310—324.
13. Toader, Gh., *Convex sequences and Fourier series*, Univ. "Babeş-Bolyai", Preprint 7(1987), 35—42.
14. Toader, Gh., *On the convexity of high order of sequences*, Publ. l'Inst. Math., **43** (**57**) (1988), 35—40.
15. Young, W. H., *On the Fourier series of bounded functions*, Proc. London Math. Soc. **12** (2) (1913), 41—70.
16. Žuk, V. V., *Approximation of periodical functions*, Univ. Leningrad, 1982 (Russian).

Received 15.X.1989

I. Ž. Milovanović
M.A. Kovačević
University of Niš
Faculty of Electronic Engineering
Department of Mathematics
P.O. Box 73, 18000 Niš
Yugoslavia

Gh. Toader
Polytechnic Institute
Department of Mathematics
3400 Cluj-Napoca
România