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APPROXIMATE SET FUNCTION MEASURABILITY DECOMPOSITION AND RANGE UNION INCLUSION

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Abstract. Suppose that U is a set, F is a field of subsets of U, $r(F)$ is the set of all functions from F into $\exp(\mathbb{R})$, $AB(\mathbb{R})(F)$ is the set of all real-valued bounded finitely additive functions defined on F and $AB(\mathbb{R})(F)^+$ is the set of all nonnegative-valued elements of $AB(\mathbb{R})(F)$. For each μ in $AB(\mathbb{R})(F)^+$, let M_μ denote the set of all elements of $r(F)$ that are μ-measurable (see "Fields of Sets, Set Functions, Set Function Integrals and Finite Additivity", Internat. J. Math. & Math. Sci., vol. 7, no. 2, 1984, pp. 209–233), and if $W \subseteq \mathbb{R}$, let $M_\mu(W)$ denote the set of all elements of M_μ with range union $\subseteq W$. Given α in $r(F)$, a positive integer n and a sequence $\{\mu_k\}_{k=1}^n$ of elements of $AB(\mathbb{R})(F)^+$, the measurability of α with respect to $\int \min\{\mu_1, \ldots, \mu_n\}$ vs. the respective measurability with respect to each μ_k, $k = 1, \ldots, n$, of each of a certain sequence of "decomposition elements" $\{\alpha_k\}_{k=1}^n$ for α, is discussed.

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1. Introduction. Suppose that U is a set, F is a field of subsets of U, $r(F)$ is the set of all functions from F into $\exp(\mathbb{R})$, $AB(\mathbb{R})(F)$ is the set of all real-valued bounded finitely additive functions defined on F and $AB(\mathbb{R})(F)^+$ is the set of all nonnegative-valued elements of $AB(\mathbb{R})(F)$. For each μ in $AB(\mathbb{R})(F)^+$, let A_μ denote $\{\eta : \eta \text{ in } AB(\mathbb{R})(F), \eta \text{ absolutely continuous with respect to } \mu\}$, M_μ denote the set of all elements of $r(F)$ that are μ-measurable (see section 2), and if $W \subseteq \mathbb{R}$, let $M_\mu(W)$ denote the set of all elements of M_μ with range union $\subseteq W$.

In various papers [1 through 3, 5 through 13], functions, either from F into $\{0, 1\}$ or from F into $\exp(\{0, 1\})$, henceforth called zero-one set functions, enter nontrivially into such matters as set function integrability, absolute continuity and uniform absolute continuity. In [7] the following measurability decomposition characterization theorem was shown, zero-one set functions figuring prominently in the proof :

THEOREM 1.A. 1. *Suppose that α is in $r(F)$, n is a positive integer and $\{\mu_k\}_{k=1}^{n}$ is a sequence of elements of $AB(\mathbb{R})(F)^+$. The following two statements are equivalent:*

1) *If* $\eta = \int \min\{\mu_1, \ldots, \mu_n\}$ *(again, see section 2), then α is in M_η,*

and

2) *for each $k = 1, \ldots, n$, there is α_k in M_{μ_k} such that for each I in F,* $\alpha(I) \subseteq \sum_{k=1}^{n} \alpha_k(I)$.

Now, in our proof in [7] of the above theorem, for each $k = 1, \ldots, n$ and I in F, $\alpha_k(I)$ is either $\alpha(I)$ or $\{0\}$, so that if x is in the range union of α_k and not in the range union of α, then $x = 0$. The question that we treat in this paper is how to have a somewhat more particular measurability decomposition characterization theorem in which the range unions of the α_ks are subsets of the range union of α, except for possibly one closure element. We prove the following "approximate" analogue of Theorem 1.A.1., again relying heavily on zero-one set functions:

THEOREM 4.1. *Assume the hypothesis of Theorem 1.A.1, with η as indicated. Let W denote the range union of α. The following statements are true.*

1) *Suppose that W is bounded. Then the following statements are equivalent*

a) *α is in M_η, and*

b) *there is a continuous function f from $\mathbb{R}^n$ into $\mathbb{R}$ such that $f([W \cup \{\sup W\}]^n) \subseteq W \cup \{\sup W\}$ and for $k = 1, \ldots, n$ an element α_k of $M_{\mu_k}(W \cup \{\sup W\})$ such that $\alpha = f(\alpha_1, \ldots, \alpha_n)$.*

2) *Suppose that W is not bounded. Then the following statements are equivalent*

a) *α is in M_η, and,*

b) *if $0 < c$, then there is a continuous function f from $\mathbb{R}^n$ into $\mathbb{R}$ such that $f(W^n) \subseteq W$ and for $k = 1, \ldots, n$ an element α_k of $M_{\mu_k}(W)$ such that if β is the function with domain F given by*

$$\beta(I) = \begin{cases} 0 & \text{if } \alpha(I) \subseteq f(\alpha_1(I), \ldots, \alpha_n(I)) \\ 1 & \text{otherwise} \end{cases}$$

then

$$\int_U L(\beta\eta)(I) < c,$$

where L is the sum supremum functional (see section 2).

2. Preliminary remarks. We begin with a brief discussion of some basic notions.

If V is in F, then the statement that D is a subdivision of V means that D is a finite collection of mutually exclusive sets of F whose union is V.

The statement that E is a refinement of D, denoted by $E \ll D$, means that for some V of F, each of D and E is a subdivision of V and each element of E is a subset of some element of D.

All set function integrals in this paper shall be refinement-wise limits of the appropriate sums. Thus, if α is a function from F into $\exp(\mathbb{R})$, then the statement that K is an integral of α on V means that V is in F, K is in $\mathbb{R}$, and if $0 < c$, then there is a subdivision D of V such that if $E \ll D$ and b is a function with domain E such that for each I in E, $b(I)$ is in $\alpha(I)$, i.e., b is an α-function on E, then

$$|[\sum_E b(I)] - K| < c;$$

K is unique and is denoted variously by $\int_V \alpha(I)$, $\int_V \alpha$, etc. We refer the reader to [9] for a detailed discussion of this and related matters, including Kolmogoroff's [14] notion of differential equivalence and certain of its more immediate consequences. We also refer the reader to [9] for various further notions, together with their pertinent notations, that we shall use in this paper, such as $\sum$-boundedness, sum supremum functional L and sum infimum functional G, and for certain fundamental sum-refinement inequalities and the consequent integral existences that arise from them. Although we also refer the reader to [4] and [9] for the notion of set function measurability, we shall, in section 3, state the definition in somewhat streamlined form in terms of zero-one set functions and give some basic measurability facts.

3. Absolute continuity and set function measurability. If $a \leqslant b$, then we let $[a; b] = \{x : a \leqslant x \leqslant b\}$.

We state a theorem which is an accumulation of some absolute continuity, mutual singularity and integrability properties. First a definition.

Definition 3.A.1 (see [9]). If each of ξ and η is in $AB(\mathbb{R})(F)^+$, then $\lambda(\xi, \eta)$ denotes the function with domain F given by

$$\lambda(\xi, \eta)(V) = \sup\left\{\int_V \min\{\xi(I), K\eta(I)\} : 0 \leqslant K\right\}.$$

THEOREM 3.A.1 (see [2,9]). *Suppose that each of ξ and μ is in $AB(\mathbb{R})(F)^+$. Then the following statements are true*

1) *Each of $\lambda(\xi, \mu)$ and $\xi - \lambda(\xi, \mu)$ is in $AB(\mathbb{R})(F)^+$ and $\lambda(\xi, \mu)$ is in A_μ.*

2) $\int_U \min\{\xi(I) - \lambda(\xi, \mu)(I), \mu(I)\} = 0.$

3) *If* $\rho = \int \min\{\xi, \mu\}$, *then* $\lambda(\xi, \mu) = \lambda(\xi, \rho)$.

4) *If α is in $r(F)$, then the following statements are true (actually, each is part of a characterization theorem)*:

a) *If for some $M \geqslant 0$, range union' $\alpha \subseteq [0\,;\,M]$, ξ is in A_μ, $\int_U \alpha(I)\mu(I)$ exists and is 0, then $\int_U \alpha(I)\xi(I)$ exists and is 0.*

b) *If for some $M \geqslant 0$, range union $\alpha \subseteq [-M\,;\,M]$, ξ is in A_μ and $\int_U \alpha(I)\mu(I)$ exists, then $\int_U \alpha(I)\xi(I)$ exists.*

We now give, with the aid of zero-one set functions, a somewhat more condensed definition of set function measurability than that given in [4]. First, some preliminary definitions.

Definition 3.2. If α is in $r(F)$ and $W \subseteq \mathbb{R}$, then $\beta(\alpha, W)$ denotes the function with domain F given by

$$\beta(\alpha, W)(I) = \begin{cases} 0 \text{ if } \alpha(I) \subseteq W \\ 1 \text{ otherwise} \end{cases}$$

Definition 3.3. If each of α and δ is in $r(F)$, then $\beta(\alpha, \delta)$ denotes the function with domain F given by

$$\beta(\alpha, \delta)(I) = \beta(\alpha, \delta(I))(I).$$

Definition 3.A.4. (see [Boll]). If μ is in $AB(\mathbb{R})(F)^+$, then the statement that α is μ-measurable means that α is in $r(F)$ and if $p \leqslant 0 \leqslant q$, then

$$\int_U \max\{\min\{\alpha(I),\, q\},\, p\}\mu(I)$$

exists and, for $p \leqslant 0 \leqslant q$,

$$\int_U L(\beta(\alpha,\, [p\,;\,q])\mu)(I) \to 0 \text{ as } \min\{|p|,\, q\} \to \infty.$$

Let us note that if μ is in $AB(\mathbb{R})(F)^+$ α is in $r(F)$, and range union of α is bounded, then α is in M_μ iff $\int_U \alpha(I)\mu(I)$ exists.

THEOREM 3.A.2 [4]. *If f is a function from $\mathbb{R}^N$ into $\mathbb{R}$, then f is continuous iff for each μ in $AB(\mathbb{R})(F)^+$ and sequence $\{\alpha_k\}_{k=1}^N$ of elements of M_μ, $f(\alpha_1, \ldots, \alpha_N)$ is in M_μ.*

THEOREM 3.1. *If each of η and ζ is in $AB(\mathbb{R})(F)^+$, then $M_\eta \cap M_\zeta = M_{\eta+\zeta}$ and, if ζ is in A_η, then $M_\eta \subseteq M_\zeta$.*

Indication of proof. Suppose that α is in $r(F)$ and $p \leqslant 0 \leqslant q$. Then $\int_U \max\{\min\{\alpha(I),\, q\},\, p\}(\eta(I) + \zeta(I))$ exists iff for $\nu = \eta$ or ζ, $\int_U \max\{\min\{\alpha(I),\, q\},\, p\}\nu(I)$ exists. Also,

$$\max\left\{\int_U L(\beta(\alpha,\, [p\,;\,q])\eta)(I)\right\},\ \max\left\{\int_U L(\beta(\alpha,\, [p\,;\,q])\zeta)(I)\right\} \leqslant$$

$$\leqslant \int_U L(\beta(\alpha,\, [p\,;\,q])(\eta+\zeta))(I) \leqslant \int_U L(\beta(\alpha,\, [p\,;\,q])\eta)(I) + $$

$$+ \int_U L(\beta(\alpha,\, [p\,;\,q])\zeta)(I).$$

The final assertion of the theorem is an easy consequence of Theorem 3.A.1 and routine considerations of absolute continuity.

LEMMA 3.A.1 [5]. *If each of δ and ω is in $r(F)$ and is $\sum$-bounded on U with respect to D, then $\delta + \omega$ is $\sum$-bounded on U with respect to D and for each V in F,*

$$G(\delta)(V) + G(\omega)(V) \leqslant G(\delta + \omega)(V) \leqslant L(\delta + \omega)(V) \leqslant L(\delta)(V) + L(\omega)(V),$$

so that

$$L(\delta + \omega)(V) - G(\delta + \omega)(V) \leqslant L(\delta)(V) - G(\delta)(V) + L(\omega)(V) - G(\omega)(V).$$

We now state two lemmas whose routine proofs we leave to the reader.

LEMMA 3.2. *If each of Y and Z is in $r(F)$, $\sum$-bounded on U with respect to D, and for each I in F, $V(I) \subseteq Z(I)$, then for each V in F,*

$$G(Z)(V) \leqslant G(Y)(V) \leqslant L(Y)(V) \leqslant L(Z)(V).$$

LEMMA 3.3. *If each of α and ω is in $r(F)$, $0 \leqslant q$, range union $\alpha \subseteq [-q\,;\,q]$*, range union $\omega \subseteq \mathbb{R}^+$, and *$\omega$ is $\sum$-bounded on U with respect to D, then $\alpha\omega$* is $\sum$-bounded on U with respect to D and if Q is L or G, *then for each V in F,*

$$|Q(\alpha\omega)(V)| \leqslant qL(\omega)(V).$$

THEOREM 3.2. *If μ is in $AB(\mathbb{R})(F)^+$, α is in $r(F)$, $0 \leqslant q$ and range union $\alpha \subseteq [-q\,;\,q]$, then the following two statements are equivalent*

1) $\int_U \alpha(I)\mu(I)$ *exists, and*

2) *if $0 < c$, then there is γ in $r(F)$ with range union $\subseteq [-q\,;\,q]$ such that $\int_U \gamma(I)\mu(I)$ exists and $\int_U L(\beta(\alpha,\, \gamma)\mu)(I) < c$.*

Proof. It is obvious that 1) implies 2).

Now suppose that 2) is true and $0 < c$. Then there is δ in $r(F)$ with range union $\subseteq [-q; q]$ such that $\int_U \delta(I)\mu(I)$ exists and $\int_U L(\beta(\alpha, \delta)\mu)(I) < c/(4q+1)$. Let $\beta = \beta(\alpha, \delta)$. Note that if I is in F, then $[\alpha(1-\beta)](I) \subseteq [\delta(1-\beta)](I)$. Thus

$$\int_U [L(\alpha\mu)(I) - G(\mu\alpha)(I)] \leq$$

$$\leq \int_U [L(\alpha(1-\beta)\mu + \alpha\beta\mu)(I) - G(\alpha(1-\beta)\mu + \alpha\beta\mu)(I)] \leq$$

$$\leq \int_U [L(\alpha(1-\beta)\mu)(I) - G(\alpha(1-\beta)\mu)(I)] + \int_U [L(\alpha\beta\mu)(I) - G(\alpha\beta\mu)(I)] \leq$$

$$\leq \int_U [L(\delta(1-\beta)\mu)(I) - G(\delta(1-\beta)\mu)(I)] + 2q\int_U L(\beta\mu)(I) \leq$$

$$\leq \int_U [L(\delta\mu - \delta\beta\mu)(I) - G(\delta\mu - \delta\beta\mu)(I)] + 2qc/(4q+1) \leq$$

$$\leq \int_U [L(\delta\mu)(I) - G(\delta\mu)(I)] + \int_U [L(-\delta\beta\mu)(I) - G(-\delta\beta\mu)(I)] + c/2 \leq$$

$$\leq 0 + 2q\int_U L(\beta\mu)(I) + c/2 \leq 2qc/(4q+1) + c/2 < c.$$

Therefore $\int_U [L(\alpha\mu)(I) - G(\alpha\mu)(I)] = 0$, so that $\int_U \alpha(I)\mu(I)$ exists.

Therefore 2) implies 1).

Therefore 1) and 2) are equivalent.

THEOREM 3.3. *If μ is in $AB(\mathbb{R})(F)^+$ and α is in $r(F)$, then the following two statements are equivalent*

1) *α is in M_μ, and*

2) *if $0 < c$, then there is γ in M_μ such that $\int_U L(\beta(\alpha, \gamma)\mu)(I) < c$.*

Proof. It is obvious that 1) implies 2).

Now suppose that 2) is true. Suppose that $p \leq 0 \leq q$. Let $K = \max\{|p|, |q|\}$. Suppose that $0 < c$. There is γ in M_μ such that $\int_U L(\beta(\alpha, \gamma)\mu)(I) < c$. If I is in F, then $\beta(\max\{\min\{\alpha, q\}, p\}, \max\{\min\{\gamma, q\}, p\})(I) \leq \beta(\alpha, \gamma)(I)$, so that $\int_U L(\beta(\max\{\min\{\alpha, q\}, p\}, \max\{\min\{\gamma, q\}, p\})\mu)(I) \leq \int_U L(\beta(\alpha, \gamma)\mu)(I) < c$. Furthermore, range union $\max\{\min\{\alpha, q\}, p\} \cup \max\{\min\{\gamma, q\}, p\} \subseteq [-K; K]$ and $\int_U \max\{\min\{\gamma(I), q\}, p\}\mu(I)$ exists. Therefore by Theorem 3.2, $\int_U \max\{\min\{\alpha(I), q\}, p\}\mu(I)$ exists.

We now show that for $p \leq 0 \leq q$, $\int_U L(\beta(\alpha, [p; q])\mu)(I) \to 0$ as $\min\{|p|, q\} \to \infty$. Suppose that $0 < c$ and $p \leq 0 \leq q$. Again, there is γ in M_μ such that

$$\int_U L(\beta(\alpha, \gamma)\mu)(I) < c/3.$$

Let $\beta_1 = \beta(\alpha, [p; q])$, $\beta_2 = \beta(\alpha, \gamma)$ and $\beta_3 = \beta(\gamma, [p; q])$. Clearly, if I is in F and $\beta_1(I) = 1$ and $\beta_2(I) = 0$, then $\beta_3(I) = 1$. Therefore, if I is in F, then

$$\beta_1(I)(1 - \beta_2(I)) \leq \beta_3(I).$$

Therefore if V is in F, then
$L(\beta(\alpha, [p; q])\mu)(V) = L(\beta_1(1-\beta_2)\mu + \beta_1\beta_2(1-\beta_3)\mu + \beta_1\beta_2\beta_3\mu)(V) \leq L(\beta_1(1-\beta_2)\mu)(V) + L(\beta_1\beta_2(1-\beta_3)\mu)(V) + L(\beta_1\beta_2\beta_3\mu)(V) \leq L(\beta_3\mu)(V) + L(\beta_2\mu)(V) + L(\beta_3\mu)(V)$. There is p' and q' such that $p' \leq 0 \leq q'$ and if $p \leq p' \leq 0 \leq q' \leq q$, then $\int_U L(\beta_3\mu)(I) < c/3$, so that $\int_U L(\beta(\alpha, [p; q])\mu)(I) < c/3 + c/3 + c/3 = c$.

Therefore 2) implies 1).

Therefore 1) and 2) are equivalent.

4. An approximate decomposition theorem. We begin with a lemma which is an extraction and simplification of the main parts of the proof of the principal theorem of [7].

LEMMA 4.1. *Suppose that $\{\mu_k\}_{k=1}^n$ is a sequence of elements of $AB(\mathbb{R})(F)^+$ and $2 \leq n$. Then there is a sequence $\{\beta_k\}_{k=1}^n$ of functions from F into $\{0, 1\}$ such that the following existence and equality assertions hold:*

i) *For each I in F, $\sum_{k=1}^n \beta_k(I) = 1$,*

ii) *for $k = 1, \ldots, n-1$, $\int_U \beta_k(I)\mu_k(I) = 0$, and*

iii) $\int \beta_n \mu_n = \lambda\left(\mu_n, \int \min\{\mu_1, \ldots, \mu_n\}\right)$.

Proof. Suppose that each of ξ and η is in $AB(\mathbb{R})(F)^+$. Let $\lambda = \lambda(\xi, \eta)$, which is $\lambda\left(\xi, \int \min\{\xi, \eta\}\right)$. By Theorem 3.A.1,

$$\int_U \min(\xi(I) - \lambda(I), \eta(I)\} = 0.$$

There is a function β with domain F such that if I is in F, then

$$\beta(I) = \begin{cases} 1 & \text{if } \eta(I) \leqslant \xi(I) - \lambda(I), \\ 0 & \text{otherwise} \end{cases}$$

If $D \ll \{U\}$, then

$$\max\{\sum_D \beta(I)\mu(I), \sum_D (1 - \beta(I))(\xi(I) - \lambda(I))\} \leqslant \sum_D \min\{\xi(I) - \lambda(I), \eta(I)\}.$$

Therefore we have the following existence and equality

$$0 = \int_U \beta(I)\eta(I) = \int_U (1 - \beta(I))(\xi(I) - \lambda(I)).$$

By Theorem 3.A.1,

$$0 = \int_U \beta(I)\lambda(I),$$

so that, since

$$(1 - \beta)\xi = (1 - \beta)(\xi - \lambda) + (1 - \beta)\lambda,$$

it follows that we have the following existence and equality

$$\int (1 - \beta)\xi = \lambda.$$

We now continue by induction. First, we note that the first paragraph establishes the conclusion statement of the lemma for the case $n = 2$.

Let us now suppose that m is a positive integer $\geqslant 2$ such that for $n = m$, the statement of the lemma holds. Suppose that $\{\mu_k\}_{k=1}^{m+1}$ is a sequence of elements of $AB(\mathbb{R})(F)^+$. Let

$$\xi = \mu_{m+1} \text{ and } \eta = \int \min\{\mu_1, \ldots, \mu_m\}.$$

Again, by the first paragraph, there is a function β from F into $\{0, 1\}$ such that we have the following existences and equalities

$$0 = \int_U \beta(I)\eta(I) \text{ and } \int (1 - \beta)\xi = \lambda,$$

i.e.,

$$0 = \int_U [\beta(I) \int_I \min\{\mu_1(J), \ldots, \mu_m(J)\}] \text{ and } \int (1 - \beta)\mu_{m+1} =$$

$$= \lambda(\mu_{m+1}, \int \min\{\mu_1, \ldots, \mu_m\}) = \lambda(\mu_{m+1}, \int \min\{\mu_1, \ldots, \mu_{m+1}\}).$$

By our inductive assumption there is a sequence $\{\beta_k\}_{k=1}^m$ of functions from F into $\{0, 1\}$ such that with respect to $\{\mu_k\}_{k=1}^m$ the conditions of the conclusion of Lemma 4.1 hold. For each $k = 1, \ldots, m$, let $\beta_k' = \beta\beta_k$, and let $\beta_{m+1}' = 1 - \beta$. The following are true

i') For each I in F, $\sum_{k=1}^{m+1} \beta_k'(I) = \left[\sum_{k=1}^{m} \beta(I)\beta_k(I)\right] + (1 - \beta(I)) =$
$= \beta(I)\left[\sum_{k=1}^{m} \beta_k(I)\right] + (1 - \beta(I)) = \beta(I) \cdot 1 + 1 - \beta(I) = 1,$

ii') for $k = 1, \ldots, m - 1$ and I in F, $\beta_k'(I)\mu_k(I) = \beta(I)\beta_k(I)\mu_k(I)$, so that $\int_U \beta_k'(I)\mu_k(I) = 0$, and by

Theorem 3.A.1, since $0 = \int_U \beta(I)\eta(I)$, $0 = \int_U \beta(I)\lambda\left(\mu_m, \int \min\{\mu_1, \ldots, \mu_m\}\right)(I) =$
$= \int_U \beta(I)\beta_m(I)\mu_m(I) = \int_U \beta_m'(I)\mu_m(I)$, and

iii') for each V in F, $\lambda\left(\mu_{m+1}, \int \min\{\mu_1, \ldots, \mu_{m+1}\}\right)(V) = \lambda(\xi, \eta)(V) =$
$= \int_V (1 - \beta(I))\xi(I) = \int_V \beta_{m+1}'(I)\mu_{m+1}(I).$

This establishes the lemma.

Observation 4.1. If x is in $\mathbb{R}$, $r \leqslant 0 \leqslant s$ and c is in $\{0, 1\}$, then

$$c\max\{\min\{x, s\}, r\} = \max\{\min\{cx, s\}, r\}.$$

LEMMA 4.2. *Suppose that* ρ *is in* $AB(\mathbb{R})(F)^+$, α *is in* $r(F)$ *and for each* $\{p, q\}$ *such that* $p \leqslant 0 \leqslant q$, $\int_U \max\{\min\{\alpha(I), q\}, p\}\rho(I)$ *exists and is* 0. *Then* α *is in* M_ρ.

Indication of proof. The integral existence part of the measurability assertion is immediate. The remainder of the measurability condition follows from the equations

$$\int_U \max\{\min\{\alpha(I), s\}, 0\}\rho(I) = 0 = \int_U \max\{\min\{\alpha(I), 0\}, r\}\rho(I), \quad r \leqslant 0 \leqslant s;$$

we leave the details to the reader.

LEMMA 4.3. *Suppose that* ρ *is in* $AB(\mathbb{R})(F)^+$, γ *is in* $r(F)$ *and* β *is a function from* F *into* $\{0,1\}$ *such that* $\int_U \beta(I)\rho(I)$ *exists and is* 0. *Then* $\beta\gamma$ *is in* M_ρ.

Indication of proof. If I is in F, x is in $\gamma(I)$ and $p \leqslant 0 \leqslant q$, then, by Observation 4.1.

$$\max\{\min\{\beta(I)x,\ q\},\ p\} = \beta(I)\max\{\min\{x,\ q\},\ p\};$$

this routinely implies that

$$\int_U \max\{\min\{\beta(I)\gamma(I),\ q\},\ p\}\rho(I)$$

exists and is 0.

Therefore, by Lemma 4.2, $\beta\gamma$ is in M_ρ.

Observation 4.2. Suppose that β is a function from F into $\{0,1\}$ and ξ is in an element of $AB(\mathbb{R})(F)^+$ such that $\int_U \beta(I)\xi(I)$ exists. Let $\nu = \int \beta\xi$. Then $\xi - \nu$ is in $AB(\mathbb{R})(F)$ and, clearly, $\beta = \beta\beta$, so that, if V is in F, then

$$\nu(V) = \int_V \beta(I)\xi(I) = \int_V \beta(I)\beta(I)\xi(I) = \int_V \Big[\beta(I)\int_I \beta(J)\xi(J)\Big] = \int_V \beta(I)\nu(I),$$

so that

$$\int_V \beta(I)(\xi(I) - \nu(I)) = \int_V \beta(I)\xi(I) - \int_V \beta(I)\nu(I) = \nu(V) - \nu(V) = 0.$$

LEMMA 4.4. *Assume the hypothesis of Theorem 1.A.1. with* η *as given* α *in* M_η, W = *range union of* α *and* H *in* $\mathbb{R}$. *Assume* $\{\beta_k\}_{k=1}^n$ *as in the conclusion of Lemma 4.1. Then, for* $k = 1, \ldots, n$, $\beta_k\alpha + (1-\beta_k)H$ *is in* $M_{\mu_k}(W \cup \{H\})$. *Furthermore, there is a function* k *from* F *into* $\{1, \ldots, n\}$ *such that if* I *is in* F, *then* $k(I)$ *is the only* $k' = 1, \ldots, n$, *such that* $\beta_{k'}(I) = 1$, *so that if* $k = 1, \ldots, n$, *then*

$$\beta_k(I)\alpha(I) + (1-\beta_k(I))H = \begin{cases}\{H\} & \text{if } k \neq k(I)\\ \alpha(I) & \text{if } k = k(I),\end{cases}$$

so that if Q *is max or min, then*

$$Q\{\beta_1(I)\alpha(I) + (1-\beta_1(I))H,\ \ldots,\ \beta_n(I)\alpha(I) + (1-\beta_n(I))H\} = Q\{\{H\},\ \alpha(I)\}.$$

Proof. If $k = 1, \ldots, n-1$, then, by Lemma 4.3, $\beta_k\alpha$ is in M_{μ_k}. We now consider n. From Theorem 3.1 it follows that α is in M_λ, where $\lambda = \lambda(\mu_n, \eta)$. By Observation 4.2, $\lambda = \int \beta_n\lambda$, so that from Theorem 3.A.2, $\beta_n\alpha$ is in M_λ. Again, by Observation 4.2, since $\int \beta_n\mu_n = \lambda$, it follows

that $\int_U \beta_n(I)(\mu_n(I) - \lambda(I))$ exists and is 0, so that by Lemma 4.3, $\beta_n\alpha$ is in $M_{\mu_n-\lambda}$. Therefore, by Theorem 3.1, $\beta_n\alpha$ is in $M_{\lambda+\mu_n-\lambda} = M_{\mu_n}$.

Now, suppose that $k = 1, \ldots, n$. It follows from Theorem 3.A.2 that $\beta_k\alpha + (1-\beta_k)$ is in M_{μ_k}. Suppose that I is in F. If $\beta_k(I) = 1$, then $\beta_k(I)\alpha(I) + (1-\beta_k(I))H = \alpha(I)$; if $\beta_k(I) = 0$, then $\beta_k(I)\alpha(I) + (1-\beta_k(I))H = \{H\}$. Therefore $\beta_k\alpha + (1-\beta_k)H$ is in $M_{\mu_k}(W \cup \{H\})$.

The final assertion of the lemma is almost immediate and we leave the details to the reader.

We now prove Theorem 4.1, as stated in the introduction.

Proof of Theorem 4.1. We first show that in case 1), b) implies a). So assume that 1) and b) of 1) are true. Clearly, for $k = 1, \ldots, n$, η is in A_{η_k}, so that by Theorem 3.1, α_k is in M_η, so that by Theorem 3.A.2, $f(\alpha_1, \ldots, \alpha_n)$ is in M_η, so that, obviously, α is in M_η.

We next show that in case 2), b) implies a). So assume that 2) and b) of 2) are true. For "each f that arises", by Theorem 3.A.2. $f(\alpha_1, \ldots, \alpha_n)$ is in M_η, so that by Theorem 3.3, α is in M_η.

Therefore, in each of cases 1) and 2), b) implies a).

We now show that in each of cases 1) and 2), a) implies b).

First, there is a sequence $\{\beta_k\}_{k=1}^n$ satisfying the conclusion of Lemma 4.1., so that for each H in $\mathbb{R}$, $\{\alpha\beta_k + (1-\beta_k)H\}_{k=1}^n$, satisfies the conclusion of Lemma 4.4.

Now suppose that 1) is true. Let $H = \sup W$. Let f denote the function from $\mathbb{R}^n$ into $\mathbb{R}$ given by

$$f(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}.$$

f is continuous and, if $S \subseteq \mathbb{R}$, then $f(S^n) \subseteq S$. If I is in F, then

$$\min\{\beta_1(I)\alpha(I) + (1-\beta_1(I))H,\ \ldots,\ \beta_n(I)\alpha(I) + (1-\beta_n(I))H\} = \min\{\{H\},\ \alpha(I)\} = \alpha(I).$$

Therefore, if 1) is true, then b) implies a).

Now suppose 2) is true. W.l.o.g., assume that W is unbounded above. Suppose that $0 < c$. There is p and H such that $p \leqslant 0 \leqslant H$, H is in W and

$$\int_U L(\beta(\alpha,\ [p;\ H])\eta)(I) < c.$$

As before, let f denote the function from $\mathbb{R}^n$ into $\mathbb{R}$ given by

$$f(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}.$$

Again f is continuous, and, if $S \subseteq \mathbb{R}$, then $f(S^n) \subseteq S$. If I is in F, then

$$\min\{\beta_1(I)\alpha(I) + (1-\beta_1(I))H,\ \ldots,\ \beta_n(I)\alpha(I) + (1-\beta_n(I))H\} = \min\{\{H\},\ \alpha(I)\},$$

so that if

$$\beta(\alpha,\ f(\beta_1\alpha + (1-\beta_1)H, \ldots, \beta_n\alpha + (1-\beta_n)H))(I) = 1,$$

then $\alpha(I)$ is not a subset of min $\{\alpha(I), \{H\}\}$, so that, clearly $\alpha(I)$ is not a subset of $[p\,;\ H]$, so that

$$\beta(\alpha,\ [p\,;\ H])(I) = 1.$$

Therefore

$$\beta(\alpha,\ f(\beta_1\alpha + (1-\beta_1)H, \ldots, \beta_n\alpha + (1-\beta_n)H))(I) \leqslant \beta(\alpha,\ [p\,;\ H])(I).$$

Therefore

$$\int_U L(\beta(\alpha,\ f(\beta_1\alpha + (1-\beta_1)H, \ldots, \beta_n\alpha + (1-\beta_n)H))\eta(I) \leqslant$$

$$\leqslant \int_U L(\beta(\alpha,\ [p\,;\ H])\eta)(I) < c.$$

For the case in which W is unbounded below, we use "max" instead of "min"; we leave the details to the reader.

Therefore, if 2) is true, then b) implies a).

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