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A NEW SUBCLASS OF QUASI CONVEX FUNCTIONS

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Abstract. In this paper, we introduce the class $C^*(\beta, \gamma)$ of quasi-convex functions of order β and type γ. It is shown that the functions in this class are closse-to-convex and hence univalent. Coefficient result, Libera's and Livingston's problems in the generalized form and some other basic properties of this class are studied.

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1. Introduction. Let S be the class of functions f, given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic and univalent in $E = \{z : |z| < 1\}$. Let C, S^* and K be the classes of convex, starlike and close-to-convex functions, respectively. We say that $f \in S$ is a convex function of order β, $0 \leqslant \beta \leqslant 1$, if

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} > \beta, \quad z \in E.$$

We denote this class by $C(\beta)$.

Also $f \in S$ is a star-like function of order β, $0 \leqslant \beta \leqslant 1$, if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \beta, \quad z \in E,$$

and we denote this class of functions by $S^*(\beta)$.

Let $K(\beta, \gamma)$ denote the class of close-to-convex functions f of order β type γ which are univalent in E and satisfy

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \beta, \quad z \in E,$$

where $g \in S^*(\gamma)$ and $0 \leqslant \beta, \gamma \leqslant 1$.

We notice that $C(0) = C$, $S^*(0) = S$ and $K(0, 0) = K$. It is clear that

$$C(\beta) \subset S^*(\beta) \subset K(\beta, \gamma)$$

We now define the class $C^*(\beta, \gamma)$ of quasi-convex functions of order β type γ as following:

Definition 1.1. A function f, analytic in E and given by (1.1), is said to be quasi-convex of order β type γ if and only if there exists a function $g \in C(\gamma)$ such that

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > \beta, \quad z \in E,$$

where $0 < \beta, \gamma \leqslant 1$. We denote this class of functions by $C^*(\beta, \gamma)$. Clearly $C^*(0, 0) = C^*$, the class of quasi-convex function introduced and discussed in [7] and [8].

We shall discuss some of the properties of the class $C^*(\beta, \gamma)$. For this purpose, we need the following.

LEMMA 1.1. [5]. *Let $0 < \lambda \leqslant 1$ and let f be defined by*

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z z^{\frac{1}{\lambda}-2} F(z)\, dz, \tag{1.2}$$

where $F \in K(\beta, \gamma)$, $0 \leqslant \beta \leqslant 1$, $0 \leqslant \gamma \leqslant 1$. *Then* $f \in K(\beta, \sigma)$ *where*

(i) *If* $0 < \lambda \leqslant \frac{1}{2}$ and $\frac{\lambda}{2(\lambda-1)} \leqslant \gamma < 1$, *then*

$$\sigma = \sigma_1 = [2\lambda\gamma + \lambda - 2 + \sqrt{(4\lambda^2\gamma^2 - 12\lambda^2\gamma + 8\lambda\gamma + 9\lambda^2 - 4\lambda + 4)}]/4\lambda) \geqslant 0$$

(ii) *If* $\frac{1}{2} < \lambda \leqslant 1$ *and* $\frac{\lambda - 1}{2\lambda} \leqslant \frac{3\lambda - \sqrt{8\lambda}}{2\lambda} \leqslant \gamma$, *then*

$$\sigma = \sigma_2 = [2\lambda\gamma + \lambda - \sqrt{4\lambda^2\gamma^2 - 12\lambda^2\gamma + 9\lambda^2 - 8\lambda}]/4\lambda \geqslant 0$$

and

(iii) *If* $\frac{1}{2} < \lambda \leqslant 1$ *and* $\frac{\lambda - 1}{2\lambda} < \frac{3\lambda - \sqrt{8\lambda}}{2\lambda} < \gamma < 1$, *then* $\sigma = \sigma_1$.

LEMMA 1.2. *Let $F \in K(\beta, \gamma)$ and let f be given as*

$$f(z) = (1 - \lambda) F(z) + \lambda z F'(z), \quad \lambda > 0 \tag{1.3}$$

Then $f \in K(\mu, \sigma)$ for $|z| < r_0$, where $\gamma \leqslant \sigma \leqslant 1$, $\beta \leqslant \mu \leqslant 1$ and r_0 is defined as

$$r_0 = \min(r_1, r_2),$$

r_1 is the least positive root of

$$[1 - (1 - 2\lambda(1 - \gamma))r][(1 - \mu) + 2\{(\beta - \mu) +$$
$$+ \lambda(\beta + \delta + \mu(1 - \gamma) - 2)\}r + (2\beta - \mu - 1)(1 - 2\lambda(1 - \delta))r^2] = 0,$$

and r_2 is the least positive root of

$$(1 - \sigma) + 2\{(\gamma - \sigma) +$$
$$+ \lambda(1 - \gamma)(\sigma - 2)\}r + (2\gamma - \sigma - 1)(1 - 2\lambda(1 - \delta))r^2 = 0.$$

The result is best possible as can be seen by $f_0(z) = (1 - \lambda) F_0(z) + \lambda z F_0(z)$, *where* $F_0(z) = f_0(z)/[1 - \lambda + \lambda z] \in K(\beta, \alpha)$ *and*

$$f_0(z) = \begin{cases} \dfrac{z(1-\gamma)(1-2\beta) + (\beta-\gamma)[1-(1-z)^{2-2\gamma}]}{(1-\gamma)(1-2\gamma)(1-z)^{2-2\gamma}}, & \gamma \neq \frac{1}{2},\ \gamma \neq 1 \\ (1-2\beta)\log(1-z) + \dfrac{2(1-\beta)z}{(1-z)}, & \gamma = \frac{1}{2} \\ 2(\beta-1)\log(1-z) + (2\beta-1)z, & \gamma = 1 \end{cases} \tag{1.4}$$

For this result, we refer to [6]

2. Main results

THEOREM 2.1. *Every quasi-convex function of order β type γ is close-to-convex function of the same order and hence univalent.*

Proof. Since $f \in C^*(\beta, \gamma)$, it implies that

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > \beta, \; g \in C(\gamma), \; z \in E.$$

This means, by using a result of Bernardi [1], that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \beta, \; g \in C(\gamma)$$

But it is known [3] that if $g \in C(\gamma)$, then $g \in S^*(\mu)$, where $\mu = \{(2\gamma - 1) + \sqrt{9 - 4\gamma + 4\gamma^2}\}/4$. Hence $\operatorname{Re} \frac{zf'(z)}{g(z)} > \beta$, $g \in S^*(\mu)$, $z \in E$.
This implies that $f \in K(\beta, \mu)$ and the proof is complete.

Remark. From the definition of $C^*(\beta, \gamma)$, we can see that

$$f \in C^*(\beta, \gamma) \Leftrightarrow zf' \in K(\beta, \gamma) \tag{2.1}$$

From (2.1) and the results for the class $K(\beta, \gamma)$ in [4], we have the following.

THEOREM 2.2. *Let $f \in C^*(\beta, \gamma)$ and be given by* (1.1). *Then*
(i) $|a_n| \leqslant \{2(3 - 2\gamma) \ldots (n - 2\gamma)[n(1 - \beta) + (\beta - \gamma)]\}/n(n)!$, for all n

ii) $$\frac{1}{r}\int_0^r \frac{(1-r)\,dr}{(1+r)^{2-2\gamma}[1+(1-2\beta)r]} \leqslant |f'(z)|$$

$$\leqslant \begin{cases} \dfrac{(1-\gamma)(1-2\beta)+(\beta-\gamma)[1-(1-r)^{2-2\gamma}r^{-1}]}{(1-\gamma)(1-2\gamma)(1-z)^{2-2\gamma}}, & \gamma \neq \dfrac{1}{2},\ \gamma \neq 1 \\ (1-2\beta)\dfrac{1}{r}\log(1-r)+\dfrac{2(1-\beta)}{r}, & \gamma = \dfrac{1}{2} \\ 2(\beta-1)\dfrac{1}{r}\log(1-r)+(2\beta-1), & \gamma = 1 \end{cases}$$

where $|z| = r$, $0 \leqslant r < 1$. The first result and the right side of the second are sharp, where the extremal function is $f_0(z)$ as defined by (1.4).

THEOREM 2.3. *Let $f \in S$ and $g \in C$. Let $\operatorname{Re}\dfrac{zf'(z)}{g(z)} > \beta$, for $z \in E$. Then $f \in C^*(\beta, 0)$ for $|z| < r_0 = \dfrac{1}{3}$. If $\beta < 3/4$, then $f \in C^*$ for $|z| < r_1 = \dfrac{1}{3-4\beta}$.*

Proof. We can write, for $z \in E$,

$$zf'(z) = g(z)\ p(z),\ \operatorname{Re} p(z) > \beta, \text{ where } g \in C.$$

Thus

$$\frac{(zf'(z))'}{g'(z)} = p(z) + \frac{g(z)}{g'(z)}p'(z),$$

from which it follows that

(2.2) $$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)} - \beta\right\} \geqslant \operatorname{Re} p(z) - \beta - \left|\frac{g(z)}{g'(z)}p'(z)\right|$$

Since g is convex in E, so we have $\operatorname{Re}\dfrac{zg'(z)}{g(z)} > \dfrac{1}{2}$, $z \in E$ and consequently from a known result in [2], we have $\operatorname{Re}\dfrac{zg'(z)}{g(z)} \geqslant (1+r)^{-1}$. This implies that

(2.3) $$\left|\frac{g(z)}{g'(z)}\right| \leqslant r(1+r)$$

Also

(2.4) $$|p'(z)| \leqslant \frac{2\operatorname{Re}[p(z)-\beta]}{1-r^2},$$

(2.5) $$\frac{1+(2\beta-1)r}{1+r} \leqslant \operatorname{Re} p(z) \leqslant \frac{1+(1-2\beta)r}{1-r}$$

see [2,9].

Using (2.3) and (2.4), we have from (2.2)

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)} - \beta\right\} \geqslant \operatorname{Re}[p(z)-\beta]\left\{1 - \frac{2r(1+r)}{1-r^2}\right\} =$$

$$= [\operatorname{Re} p(z) - \beta]\left\{\frac{1-3r}{1-r}\right\}.$$

Hence $f \in C^*(\beta, 0)$ for $|z| < \dfrac{1}{3}$.

We now prove the second part of the theorem. For $z \in E$, we have from (2.2), (2.3) and (3.4).

$$\operatorname{Re}\frac{(zf'(z))'}{g'(z)} \geqslant \operatorname{Re} p(z) - 2r(1+r)\left\{\frac{\operatorname{Re} p(z) - \beta}{1-r^2}\right\} =$$

$$= \operatorname{Re} p(z)\left[1 - \frac{2r}{1-r}\right] + \frac{2\beta r}{1-r} \geqslant$$

$$\geqslant \frac{1+(2\beta-1)r}{1+r}\left(\frac{1-3r}{1-r}\right) + \frac{2\beta r}{1-r} =$$

$$= \frac{1+4(\beta-1)r+r^2(3-4\beta)}{1-r^2} = \frac{1-(3-4\beta)r}{1+r}.$$

Hence $\operatorname{Re}\dfrac{(zf'(z))'}{g'(z)} > 0$, for $|z| < r_1 = \dfrac{1}{3-4\beta}$.

We note that, for $\beta = \dfrac{1}{2}$, $f \in C^*$, for $z \in E$.

THEOREM 2.4. *Let $0 < \lambda \leqslant 1$. Let f be defined by (1.2), where $F \in C^*(\beta, \gamma)$. Then $f \in C^*(\beta, \sigma)$ where σ is defined in Lemma 1.1.*

Proof. Let $F_0(z) = zF'(z)$ and let

(2.6) $$f_0(z) = \frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z z^{\frac{1}{\lambda}-2}F_0(z)\,dz$$

From (2.1) and Lemma 1.1, it follows that $f_0 \in K(\beta, \sigma)$. Now (2.6) can be written as

$$f_0(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z z^{\frac{1}{\lambda}-1} F'(z)\, dz, \text{ since } F_0(z) = zF'(z).$$

Integration by parts gives us

$$f_0(z) = \frac{1}{\lambda} F(z) - \frac{1}{\lambda}\left(\frac{1}{\lambda} - 1\right) z^{1-\frac{1}{\lambda}} \int_0^z z^{\frac{1}{\lambda}-2} F(z)\, dz =$$

$$= z \frac{d}{dz}\left[\left(\frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z z^{\frac{1}{\lambda}-2} F(z)\, dz\right)\right] = zf'(z),$$

and now the result follows immediately by using (2.1).

THEOREM 2.5. *Let* $F \in C^*(\beta, \gamma)$ *and* f *be defined by (1.3). Then* $f \in C^*(\mu, \sigma)$ *for* $|z| < r_0$, *where* r_0, μ, β, σ, γ *are defined as in Lemma 1.2.* This result is best possible.

Proof. From (1.3), we can write

$$zf'(z) = (1 - \lambda_1)\; zF'(z) + \lambda_1 z(zF'(z))' =$$
$$= (1 - \lambda_1) H(z) + \lambda_1 z H'(z).$$

Now, from (2.1), it follows that $H = zF' \in K(\beta, \gamma)$. Thus, from Lemma 1.2, we see that $zf' \in K(\mu, \sigma)$ for $|z| < r_0$ and consequently $f \in C^*(\mu, \sigma)$, where β, γ, σ, μ and r_0 are as given in Lemma 1.2. Equality holds for $f_0(z) = (1 - \lambda_1)F_0(z) + \lambda_1 z F_0'(z)$, where F_0 can be obtained from the condition (2.1) and the extremal function of Lemma 1.2 as defined by (1.4).

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