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ON A SPECIAL CLASS OF PARETO BICRITERIAL OPTIMIZATION PROBLEMS

NICOLAE POPOVICI
(Cluj-Napoca)

1. Preliminaries. Let us consider the following general multicriterial optimization problem

$$(1.1)\quad \begin{cases} f(x) = (f_1(x), \ldots, f_m(x)) \to \text{v. max} \\ \text{subject to } x \in \Omega \subseteq \mathbb{R}^n \end{cases}$$

One of the theoretical approaches for generating some nondominated solutions for the above problem, which appears repeatedly in the literature is based on reduction the vectorial optimization problem (1.1) to a family of scalar optimization problems. In order to describe this method let us write

$$\Lambda = \left\{ \lambda = (\lambda_1, \ldots, \lambda_m) : \lambda \in \mathbb{R}^m, \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0, (i = 1, \ldots, m) \right\}.$$

Given $\lambda \in \Lambda$ let $P(\lambda)$ denote the following problem

$$(1.2)\quad \begin{cases} F(x) = \sum_{i=1}^m \lambda_i f_i(x) \to \max \\ \text{subject to } x \in \Omega \end{cases}$$

Let $(L) = \{x \in \Omega : x \text{ solve } P(\lambda) \text{ for some } \lambda \in \operatorname{Int} \Lambda\}$ and let $M_P(f, \Omega)$ be the set of all Pareto maximum points of the vectorial objective function f on Ω, i.e. the set of all nondominated solutions of problem (1.1).

The following result is well-known (Da Cunha, Polak, 1967):

$$(L) \subseteq M_P(f, \Omega).$$

which allows to generate some nondominated solutions for (1.1) by means of the problems $P(\lambda)$, $\lambda \in \operatorname{Int} \Lambda$.

This approach works quite well for the bicriterial case. In what follows we shall use the above method for a special class of bicriterial optimization problems.

2. A certain "linear-linear fractional" bicriterial optimization problem.

Let us consider the following problem

$$
(2.1)\qquad \begin{cases} Z^{(1)} = c^{(1)}\cdot x \\ \dfrac{Z^{(2)}}{Z^{(3)}} = \dfrac{c^{(2)}\cdot x + \beta}{c^{(3)}\cdot x + \gamma} \end{cases} \rightarrow \text{v. max} \\ \text{subject to } Ax = b,\ x \geqslant 0,
$$

where $x,\ c^{(1)},\ c^{(2)},\ c^{(3)} \in \mathbb{R}^n,\ \beta,\ \gamma \in \mathbb{R},\ b \in \mathbb{R}^m$ and $A \in \mathcal{M}_{m,n}(\mathbb{R})$, on the assumption that $c^{(3)}\cdot x + \gamma > 0$ holds in the feasible set.

This problem was considered by Shyamal Chatterjee and Rabindranath Sen, who gave in [1] an algorithm for generating nondominated solutions based on a simplex procedure. In order to obtain the criterion for optimization, in [1] the authors gave two theorems ([1], Th. 2.1 and Th. 2.2).

Using the method presented in the first section of our paper let consider the problem

$$
(2.2)\qquad \begin{cases} Z = \omega Z^{(1)} + (1-\omega)\dfrac{Z^{(2)}}{Z^{(3)}} \rightarrow \max. \\ \text{subject to} \\ Ax = b,\ x \geqslant 0 \end{cases}
$$

corresponding to the $\lambda = (\omega,\ 1-\omega) \in \operatorname{Int}\Lambda$, i.e. $\omega \in]0,1[$.

In what follows in this section we shall use the notation used in [1]. Let x_B be the initial basic feasible solution and let us denote by B the basic matrix ($x_B = B^{-1}b$). Here $Z_0^{(1)} = c_B^{(1)}\cdot x_B$, $Z_0^{(2)} = C_B^{(2)}\cdot x_B + \beta$ and $Z_0^{(3)} = c_B^{(3)}\cdot x_B + \gamma$, where $c_B^{(1)}, c_B^{(2)}$ and $c_B^{(3)}$ are the vectors having their components as the coefficients associated with the basic variables in the objective function of (2.1). Let consider the following notations

$$
y_j = B^{-1}a_j,\ Z_j^{(i)} = c_B^{(i)}\cdot y_j,\ (i = 1,2,3) \text{ and } \theta = \min\{x_{B_i}/y_{ij} : y_{ij} > 0\}.
$$

In [1] the following theorem, for improving the basic feasible solution x_B is formulated

2.1. THEOREM ([1], Th. 2.1) *Given a basic feasible solution $x_B(x_B = B^{-1}b)$ for a programming problem given by (2.2), with the value of the objective function for the solution being*

$$
Z_0 = c_B^{(1)}\cdot x_B + (1-\omega)\frac{c_B^{(2)}\cdot x_B + \beta}{c_B^{(3)}\cdot x_B + \gamma}
$$

if for any column a_j in A but not in B the condition

$$
(2.3)\qquad Z_j^{(1)} + \frac{(1-\omega)}{\omega}\cdot\frac{Z_j^{(2)}}{Z_0^{(3)}} < c_j^{(1)} + \frac{(1-\omega)}{\omega}\frac{c_j^{(2)}}{Z_0^{(3)}}
$$

holds for $Z_j^{(3)} - c_j^{(3)} \geqslant 0$ and the condition

$$
(2.4)\qquad \theta(Z_j^{(1)} - c_j^{(1)}) + \frac{\omega-1}{\omega}\frac{2\theta Z_0^{(2)}(Z_j^{(3)} - c_j^{(3)}) - \theta Z_0^{(3)}(Z_j^{(2)} - c_j^{(2)})}{(Z_0^{(3)})^2 - \theta(Z_j^{(3)} - c_j^{(3)})^2} + \frac{1-\omega}{\omega}\frac{\theta^2(Z_j^{(2)} - c_j^{(2)})(Z_j^{(3)} - c_j^{(3)})}{(Z_0^{(3)})^2 - \theta(Z_j^{(3)} - c_j^{(3)})^2} \leqslant 0
$$

holds for $Z_j^{(3)} - c_j^{(3)} < 0$ and if at least one $y_{ij} > 0$, $i = 1,\dots,m$, then it is possible to obtain a new basic feasible solution $\hat{x}_B$ by replacing one of the columns in B by a_j and the new value of the objective function $\hat{Z}$ satisfies $\hat{Z} \geqslant Z_0$. Furthermore if the given basic solution is not degenerate then $\hat{Z} > Z_0$.

In the proof of this theorem there is a mistake. The conclusion of this theorem is not true, as we can see from the following

2.2. *Example.* Let us consider the following problem

$$
(2.5)\qquad \begin{cases} Z^{(1)} = -x_1 \\ \dfrac{Z^{(2)}}{Z^{(3)}} = \dfrac{x_1 + x_2 - 2x_3}{x_1 + x_2} \end{cases} \rightarrow \text{v. max} \\ \text{subject to} \\ x_1 + x_2 + x_3 = 2,\ x_2 = 1,\ x_1,\ x_2,\ x_3 \geqslant 0.
$$

We form the super-criterion

$$
Z = \omega Z^{(1)} + (1-\omega)\frac{Z^{(2)}}{Z^{(3)}}
$$

where $\omega \in]3/5,\ 2/3[$.

For the initial b.f.s. $x_B = (1,1)$, corresponding to the vertex $x^0 = (1,1,0)$ we have

$j = 3,\ y_{11} = 1,\ y_{12} = 0,\ y_{13} = 1,\ y_{21} = 0,\ y_{22} = 1,\ y_{23} = 0,$

$\theta = 1,\ Z_3^{(1)} = -1,\ Z_3^{(2)} = 1,\ Z_3^{(3)} = 1,\ c_3^{(1)} = 0,\ c_3^{(2)} = -2$ and $c_3^{(3)} = 0.$

It is clear that $Z_3^{(3)} - c_3^{(3)} \geqslant 0$ and the condition (2.3) holds for any $\omega \in]3/5, 1[$. However, for the new b.f.s. $\hat{x}_B = (1,1)$ corresponding to the vertex $\hat{x} = (0,\ 1,\ 1)$ we obtain $\hat{Z} < Z_0$ as soon as $\omega \in]0,2/3[$.

2.3. ***Remark.*** In the same paper another theorem is stated ([1], Th. 2.2). But in the proof of this theorem it is used the conclusion of Theorem 2.1 and therefore both these results are not valid. Moreover, in the proof of the second theorem ([1], Th. 2.2) the following assertion is done: "The function

$$
Z = c^{(1)}\cdot x + \frac{1-\omega}{\omega}\frac{c^{(2)}\cdot x + \beta}{c^{(3)}\cdot x + \gamma}
$$

$\omega \neq 0$, $c^{(3)} \cdot x + \gamma \neq 0$ is both pseudoconvex and pseudoconcave on a convex set $\Gamma \subseteq \mathbb{R}^n$". But this assertion is in general not true (see [4])

Some aspects concerning such problems we shall review in the next section.

3. On a hyperbolic optimization problem. Let consider the following problem

$$(3.1) \quad \begin{cases} f(x) = c \cdot x + \dfrac{e \cdot x}{d \cdot x} \to \max. \\ \text{subject to} \\ x \in \Omega = \{x \in \mathbb{R}^n : Ax \leqslant b\} \end{cases}$$

where $x, c, d, e \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathcal{M}_{m,n}(\mathbb{R})$.

Observe that the problem (2.2) is like the above problem. This problem was studied in 1969 by G. Teterev, who gave a theorem ([6], Th. 1) which is not valid. The nonvalidity of this theorem was proved by J. Hirche and Ho Khac Tan ([2], [3]). On the basis of this theorem ([6], th. 1), in 1974 I. Marușceac concluded that the function f from (3.1) is quasiconvex on the set Ω defined in (3.1) on the assumption that $\Omega \subset E_1 = \{x \in \mathbb{R}^n;\ c \cdot x \geqslant 0,\ d \cdot x > 0$ ([5], Th. 2).

In the same paper [5], using the conclusion that f is quasiconvex on the $\Omega \subset E_1$, an algorithm is constructed. In the following section we shall prove that in certain supplementary conditions this results are valid.

4. A certain "Linear-hyperbolic" bicriterial optimization problem. In order to establish sufficient conditions for f from (3.1) to be quasiconvex, in 1978 L. Lupșa gave the following

4.1. THEOREM. ([4], t.10). *If $\Omega \subset E_1$ and the following system*

$$(4.1) \quad \begin{cases} d \cdot x = 0 \\ e \cdot x > 0 \\ c \cdot x < 0,\ x \in \mathbb{R}^n \end{cases}$$

is an inconsistent system, then the function f is quasiconvex on Ω.

In what follows let consider the problem

$$(4.2) \quad \begin{cases} \left.\begin{aligned} f_1(x) &= c^{(1)} \cdot x \\ f_2(x) &= c^{(2)} \cdot x + \dfrac{e \cdot x}{d \cdot x} \end{aligned}\right\} \to \text{v. max.} \\ \text{subject to } x \in \Omega = \{x \in \mathbb{R}^n : Ax \leqslant b\} \subseteq E_1^*, \end{cases}$$

where $E_1^* = \{x \in \mathbb{R}^n : c^{(1)} \cdot x,\ c^{(2)} \cdot x \geqslant 0 \text{ and } d \cdot x > 0\}$.

Observe that there is a strong connection between the above problem and the problem (2.1). For generating some nondominated solutions for the above problem we shall use the aproach indicated in the first section. Observe that the super-criterion

$$F_\omega = \omega f_1 + (1 - \omega) f_2, \quad \omega \in]0,1[$$

is a hyperbolic function. In order to adopt the algorithm constructed in [5], the following result will be useful:

4.2. LEMMA. *Let $c^{(1)}$, $c^{(2)}$, e and d be given vectors of $\mathbb{R}^n$. If the following two systems*

$$(4.3) \quad \begin{cases} d \cdot x = 0 \\ e \cdot x > 0 \\ c^{(1)} \cdot x < 0 \end{cases}$$

and

$$(4.4) \quad \begin{cases} d \cdot x = 0 \\ e \cdot x > 0 \\ c^{(2)} \cdot x < 0 \end{cases}$$

are inconsistent systems then for any $\omega \in]0,1[$ the following system

$$(4.5) \quad \begin{cases} d \cdot x = 0 \\ e \cdot x > 0 \\ [\omega c^{(1)} + (1 - \omega) c^{(2)}] \cdot x < 0 \end{cases}$$

is an inconsistent system, too.

Proof. For every $\omega \in]0,1[$ let us put

$$M_\omega = \{x \in \mathbb{R}^n : [\omega c^{(1)} + (1 - \omega) c^{(2)}] \cdot x < 0\},$$
$$M_\omega^{(1)} = \{x \in \mathbb{R}^n : c^{(1)} \cdot x < 0\} \text{ and}$$
$$M_\omega^{(2)} = \{x \in \mathbb{R}^n : c^{(2)} \cdot x < 0\}.$$

The following conditions holds for every $\omega \in]0,1[$

$$(4.6) \quad M_\omega \subseteq M_\omega^{(1)} \cup M_\omega^{(2)}.$$

Indeed, for any $\omega \in]0,1[$ and $x^0 \in \mathbb{R}^n$, from $x^0 \notin M_\omega^{(1)} \cup M_\omega^{(2)}$ it follows that $x^0 \notin M_\omega^{(1)}$ and $x^0 \notin M_\omega^{(2)}$ and therefore $c^{(1)} \cdot x^0 \geqslant 0$ and $c^{(2)} \cdot x^0 \geqslant 0$. Then $[\omega c^{(1)} + (1 - \omega) c^{(2)}] \cdot x^0 \geqslant 0$ implies that $x^0 \notin M_\omega$. Consequently, (4.6) is proven.

Now, assume that $x^0 \in \mathbb{R}^n$ is a solution for the system (4.5). Then $x^0 \in M_\omega$ and in virtue of (4.6) it follows that $x^0 \in M_\omega^{(1)}$ or $x^0 \in M_\omega^{(2)}$. But this conclusion implies that x^0 is a solution for one of the systems (4.3) or (4.4), which represents a contradiction.

4.3. THEOREM. *If $\Omega \subset E_1^*$ and the systems (4.3) and (4.4) are both inconsistent, then for any $\omega \in]0,1[$ the function F_ω is quasiconvex on Ω.*

Proof. It follows from Theorem 4.1 and Lemma 4.2 if we take $c := \omega c^{(1)} + (1 - \omega) c^{(2)}$, $e := (1 - \omega) e$, $d := d$ and $f := F_\omega = \omega f_1 + (1 - \omega) f_2$.

4.4. *Remark.* If $\Omega \subset E_1^*$ and systems (4.3) and (4.4) are both inconsistent, we can use the algorithm constructed by I. Maruşceac in [5], for the hyperbolic programming problem

$$\begin{cases} F_\omega(x) = [\omega c^{(1)} + (1 - \omega) c^{(2)}] \cdot x + \dfrac{(1 - \omega) e \cdot x}{d \cdot x} \to \max \\ \text{subject to } x \in \Omega. \end{cases}$$

This algorithm, based on a simplex procedure consists of two phases. The problem is first approached by a simplex procedure to obtain its local maximum and then with the help of the cutting plane technique a global optimum is derived.

By means of problems (4.7), corresponding to any $\omega \in]0,1[$, we can obtain some nondominated solutions for problem (4.2) as we have seen in the first section of this paper.

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University of Cluj-Napoca
Faculty of Mathematics
3400 Cluj-Napoca
România