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NETS OF POSITIVE LINEAR FUNCTIONALS ON $C(X)$

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There are many Korovkin type results concerning the convergence of a net of positive linear functionals to a Dirac fucntional. The test set is often of the form $T \cup \{t^2 : t \in T\}$ where T is a given set of functions. In this note we present some results in this context.

Let X be a compact Hausdorff space and let $C(X)$ be the space of all real-valued continuous functions on X endowed with the sup-norm. Denote by $M_+^1(X)$ the set of all probability Radon measures on X. Let (μ_i) be a net in $M_+^1(X)$ and let $\mu \in M_+^1(X)$. Denote

$$E = \{f \in C(X) : \lim_i (\mu_i(f^2) - \mu_i^2(f)) = 0\}$$

THEOREM. a) *E is a closed subalgebra of $C(X)$.*

b) *If $\lim_i \mu_i(f) = \mu(f)$ for all $f \in E$, then every $f \in E$ is constant on* supp μ.

Proof. a) Let $\nu \in M_+^1(X)$, f, $g \in C(X)$. Then $\nu(f+a)^2 \geqslant 0$ for all $a \in R$; this yields $\nu(f^2) \geqslant \nu^2(f)$. It follows that $\nu(f+ag)^2 \geqslant \nu^2(f+ag)$ for all $a \in R$. Therefore

(1) $$(\nu(fg) - \nu(f)\,\nu(g))^2 \leqslant (\nu(f^2) - \nu^2(f))(\nu(g^2) - \nu^2(g))$$

This implies

(2) $$\lim_i (\mu_i(fg) - \mu_i(f)\,\mu_i(g)) = 0 \text{ for all } f \in E,\ g \in C(X)$$

Clearly $\lambda f \in E$ for all $\lambda \in R$ and all $f \in E$. Let now f, $g \in E$. Then

$$\mu_i(f+g)^2 - \mu_i^2(f+g) = \mu_i(f^2) - \mu_i^2(f) + \mu_i(g^2) - \mu_i^2(g) + 2(\mu_i(fg) - \mu_i(f)\mu_i(g)).$$

Using (2) we see that $f + g \in E$. Moreover,

$$\mu_i(f^4) - \mu_i^2(f^2) = [\mu_i(f^4) - \mu_i(f)\mu_i(f^3)] + \mu_i(f)[\mu_i(f^3) - \mu_i(f)\,\mu_i(f^2)] + \mu_i(f^2)[\mu_i^2(f) - \mu_i(f^2)]$$

Again by using (2) we infer that $f^2 \in E$. Since

$$fg = ((f+g)^2 - f^2 - g^2)/2$$

it follows that $fg \in E$ for all f, $g \in E$.

Hence E is a subalgebra of $C(X)$; it is easy to verify that it is closed.

b) Let $f \in E$. Then $f^2 \in E$ and thus

$$\mu(f^2) - \mu^2(f) = \lim_i (\mu_i(f^2) - \mu_i^2(f)) = 0.$$

For $x \in X$ let us denote $\varphi(x) = \mu(f - f(x))^2 =$

$$= \mu(f^2) - 2f(x)\mu(f) + f^2(x).$$

Then $\varphi \in C(X)$, $\varphi \geqslant 0$ and $\mu(\varphi) = \mu(f^2) - 2\mu^2(f) + \mu(f^2) = 0$.

It follows that $\varphi = 0$ on supp μ.

Let $x_0 \in$ supp μ. Then $\mu(f - f(x_0))^2 = 0$, i.e., $f - f(x_0) = 0$ on supp μ. Thus f is constant on supp μ and the proof is finished.

COROLLARY. *Suppose that* $T \subset C(X)$ *separates* X *and* $\lim_i \mu_i(t) = \mu(t)$, $\lim_i \mu_i(t^2) = \mu^2(t)$ *for all* $t \in T$. *The following statements are equivalent.*

(i) $\lim_i \mu_i(f) = \mu(f)$ *for all* $f \in C(X)$

(ii) μ *is a Dirac measure.*

Proof. For $t \in T$ we have $\lim_i (\mu_i(t^2) - \mu_i^2(t)) = \mu^2(t) - \mu^2(t) = 0$,

hence $T \subset E$. By the Stone-Weierstrass theorem it follows that $E = C(X)$.
(i) $\rightarrow$ (ii). Using part b) of the above theorem we infer that every $f \in C(X)$ is constant on supp μ, hence μ is a Dirac measure.
(ii) $\rightarrow$ (i). This is a well-known Korovkin type theorem. (It is connected with the Stone-Weierstrass theorem; see [1], [2] and the references given there). For the sake of completeness we present a proof.

Suppose that (i) does not hold. Let μ be concentrated at $x \in X$. Thus, for every $t \in T$ we have

(3) $$\lim_i \mu_i(t) = t(x) \text{ and } \lim_i \mu_i(t^2) = t^2(x).$$

but there exists a function $g \in C(X)$ such that $(\mu_i(g))$ is not convergent to $g(x)$.

It follows that there are an $\varepsilon > 0$ and a subnet (μ_j) of (μ_i) such that

(4) $$|\mu_j(g) - g(x)| \geqslant \varepsilon \text{ for all } j.$$

By a compactness argument there exist a subnet (μ_k) of (μ_j) and a $\nu \in M_+^1(X)$ such that

(5) $$\lim_k \mu_k(f) = \nu(f) \text{ for all } f \in C(X).$$

By the above proof that (i) $\rightarrow$ (ii), ν is a Dirac measure. Let ν be concentrated at $y \in X$. Then $\lim_k \mu_k(t) = t(y)$ for all $t \in T$. On the other hand, by (3), $\lim_k \mu_k(t) = t(x)$. Since T separates X, we see that $y = x$. Now (5) yields $\lim_k \mu_k(g) = g(x)$, which contradicts (4).

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