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ON WEAK CONTINUITY AND WEAK δ-CONTINUITY

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Changing the topology on the domain and/or the range of a function enables us to prove some theorems easily.

It is shown that a function $f:(X, \tau) \to (Y, \nu)$ is weakly δ-continuous iff $f:(X, \tau_s) \to (Y, \nu)$ is weakly continuous and f has a graph δ-closed with respect to X iff f has a graph closed in $(X \times Y, \tau_s \times \nu)$. If the range space is almost regular (regular) then the faint continuity is equivalent to almost strong θ-continuity (strong θ-continuity).

1. Introduction. We have not attempted to give the original source of all the definitions mentioned in this paper, but they may be obtained from the quoted references.

For a topological space (X, τ) and $A \subset X$, the θ-closure of A and the θ-interior of A are defined in the following way and denoted $\theta \operatorname{cl} A$ and $\theta \operatorname{int} A$ respectively (see Long and Herrington, [4]). $\theta \operatorname{cl} A = \{x \in X :$ for every open set U containing $x \bar{U} \cap A \neq \emptyset\}$, $\theta \operatorname{int} A = \{x \in A :$ there exists an open set U containing x such that $x \in U \subset \bar{U} \subset A\}$.

In a topological space (X, τ), for $A \subset X$, $\bar{A}$ and A^0 will be used for the closure of A and the interior of A respectively if there is no confusion.

A subset A of a topological space (X, τ) is called regular open (regular closed, θ-open, θ-closed) if $A = \mathring{\bar{A}}$ $(A = \bar{\mathring{A}}$, $A = \theta \operatorname{int} A$, $A = \theta \operatorname{cl} A)$ (See Long and Herrington [4]).

In a topological space (X, τ) all θ-open sets form a topology τ_θ on X Long and Herrington [4] and all regular open sets form a base for a topology τ_s on X. τ_s is called the semi-regularization topology of τ (see Raghavan [11]). It is known that $\tau_\theta \subset \tau_s \subset \tau$ and τ is semi regular iff $\tau = \tau_s$ (see Raghavan [11]), τ is almost regular iff $\tau_s = \tau_\theta$ Long and Herrington [4], τ is regular iff $\tau = \tau_\theta$ Raghavan [11].

2. Some continuity types of functions. We will not give the original definitions of the following functions, but we will give their equivalent forms.

Let $f:(X, \tau) \to (Y, \nu)$ be a function. The followings are known.
(i) f is strongly θ-continuous iff $f:(X, \tau_\theta) \to (Y, \nu)$ is Long and Herrington [3] continuous.

(ii) f is almost strongly θ-continuous iff $f:(X, \tau_\theta) \to (Y, \nu)$ is Noiri and Kang continuous [9].
(iii) f is super continuous iff $f:(X, \tau_s) \to (Y, \nu)$ is Reilly and Vamanamurty continuous [12].
(iv) f is δ-continuous iff $f:(X, \tau_s) \to (Y, \nu_s)$ is Reilly and Vamanamurty continuous [12].
(v) f is almost continuous iff $f:(X, \tau) \to (Y, \nu_s)$ is Reilly and Vamanamurty continuous [12]
(vi) f is faintly continuous iff $f:(X, \tau) \to (Y, \nu_\theta)$ is Long and Herrington continuous [4].

$f:(X, \tau) \to (Y, \nu)$ is called θ-continuous [15] (resp. weakly δ-continuous[2], weakly continuous (see Baker [2])) if for each $x \in X$ and each ν-open set V containing x, there exists an open set U such that $x \in U$ and $f(\bar{U}) \subset \bar{V}$ (resp. $f(\mathring{\bar{U}}) \subset \bar{V}$, $f(U) \subset \bar{V}$).

Remark 2.1. Clearly θ-continuity implies weak δ-continuity and weak δ-continuity implies weak continuity.

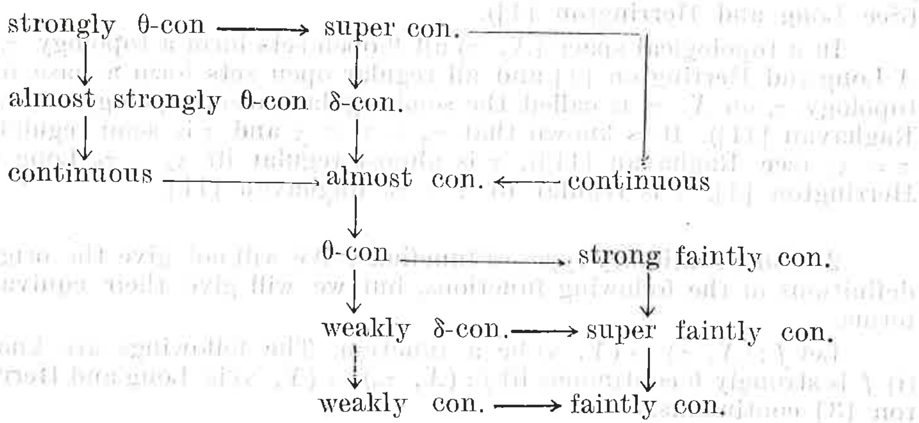
THEOREM 2.2. *Let $f:(X, \tau) \to (Y, \nu)$ be a function. The followings are valid.*
(i) *If f is θ-continuous, then $f:(X, \tau_\theta) \to (Y, \nu_\theta)$ is continuous.*
(ii) *If f is weakly δ-continuous then $f:(X, \tau_s) \to (Y, \nu_\theta)$ is continuous.*

Definition 2.3. Let $f:(X, \tau) \to (Y, \nu)$ be a function.
(i) If $f:(X, \tau_s) \to (Y, \nu_\theta)$ is continuous then f is called super faintly continuous.
(ii) If $f:(X, \tau_\theta)) \to (Y, \nu_\theta)$ is continuous then f is called strong faintly continuous.

Clearly $f:(X, \tau) \to (Y, \nu)$ is super faintly continuous (strong faintly continuous) iff for each $x \in X$ and each θ-open set V containing $f(x)$, there exists an open set U containing x such that $f(\mathring{\bar{U}}) \subset V$ ($f(\bar{U}) \subset V$).

Hence definitions of a function which is continuous when the original on X or Y are changed by semi-regularization topology or topology of θ-open sets are finished.

The following diagram is shown from Baker [2], long and Herrington [4], Noiri and Kang [9], Remark 2.1., Theorem 2.2, Def. 2.3.



Strong faint continuity does not imply weak continuity. We see this from the following example.

Example 2.4. Let $X = \{a, b, c\}$, $Y = X$, $\tau = \{\emptyset, X, \{a\}\}$ and $\nu = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ since $\nu_\theta = \{\emptyset, X\}$, every function from $(X, \tau_\theta) \to (Y, \nu_\theta)$ is continuous. But let $f: X \to Y$ be a function such that $f(b) = c$ then f is not weakly continuous.

3. Relation between weak continuity and weak δ-continuity

THEOREM 3.1. *$f:(X, \tau) \to (Y, \nu)$ is weakly δ continuous iff $f:(X, \tau_s) \to (Y, \nu)$ is weakly continuous.*

Proof. Let f be weakly δ-continuous function, $x \in X$ and V be ν-open set containing $f(x)$. There exists a τ-open set U such that $x \in U$ and $f(\mathring{\bar{U}}) \subset \bar{V} \cdot \mathring{\bar{U}} \in \tau_s$ and $x \in U \subset \mathring{\bar{U}}$. Hence $f:(X, \tau_s) \to (Y, \nu)$ is weakly continuous.

Now let $f:(X, \tau_s) \to (Y, \nu)$ be weakly continuous, $x \in X$ and V be ν-open set containing $f(x)$. There exists a τ_s-open set U such that $x \in U$ and $f(U) \subset \bar{V}$. Since U is τ_s-open and $x \in U$, there exists a regular open set U' such that $x \in U' \subset U$. We have $f(U') = f(\mathring{\bar{U}}') \subset \bar{V}$. So $f:(X, \tau) \to (Y, \nu)$ is weakly δ-continuous.

Definition 3.2. (see Baker [2]). The graph of a function $f: X \to Y$ is said to be δ-closed i with respect to X if for each $(x, y) \in X \times Y - G(f)$, there exist open sets U and V such that $x \in U \subset X$ and $y \in V \subset Y$ and $(\mathring{\bar{U}} \times V) \cap G(f) = \emptyset$.

THEOREM 3.3. *The graph of $f:(X, \tau) \to (Y, \nu)$ is δ-closed with respect to X iff $G(f)$ is closed in $(X \times Y, \tau_s \times \nu)$.*

Proof. Let the graph of $f:(X, \tau) \to (Y, \nu)$ is δ-closed with respect to X and $(x, y) \in X \times Y - G(f)$. There exist τ-open set U and ν-open set V such that $x \in U$, $y \in V$ and $(\mathring{\bar{U}} \times V) \cap G(f) = \emptyset$. Hence $(x, y) \in U \times V \in \mathring{\bar{U}} \times V \in \tau_s \times \nu$ and $(x, y) \in \mathring{\bar{U}} \times V \subset X \times Y - G(f)$. So $X \times Y - G(f)$ is open in $(X \times Y, \tau_s \times \nu)$.

Now let $G(f)$ be closed in $(X \times Y, \tau_s \times \nu)$ and $(x, y) \in X \times Y - G(f)$. Since $X \times Y - G(f)$ is open in $(X \times Y, \tau_s \times \nu)$ there exist a τ_s-open set U and a ν-open set V such that $x \in U$, $y \in V$ and $U \times V \subset X \times Y - G(f)$. Since $U \in \tau_s$ there exists a regular open set U' in (X, τ) such that $x \in U' = \mathring{\bar{U}}' \subset U$. We have $(x, y) \in \mathring{\bar{U}}' \times V) \subset U \times V$ and $(\mathring{\bar{U}}' \times V) \cap G(f) = \emptyset$. Hence the graph of f is δ-closed with respect to X.

COROLLARY 3.4. *Let $f: X \to Y$ be a function. If X is a semiregular space then weak continuity is equivalent to weak δ-continuity and $G(f)$ is closed iff $G(f)$ is δ-closed with respect to X.*

COROLLARY 3.5. *Let $f : X \to Y$ be a function. $f : (X, \tau_s)) \to (Y, \nu)$ is weakly continuous iff $f : (X, \nu_s) \to (Y, \nu)$ is weakly δ-continuous.*

Proof. It is known that $(\tau_s)_s = \tau_s$, so (X, τ_s) is semiregular Mrsevic'. Reilly and Vamanamurty [6]. The result is clear now from Corollary 3.4,

A space Y is called rim-compact (see Baker [2]) if for every y in Y and every open neighborhood V of y, there exists an open set U such that $y \in U \subset V$ and boundary of U is compact.

We give a short proof of Theorem 8 of Baker [2] below.

THEOREM 3.6. (Theorem 8 of Baker [2]). *If $f : X \to Y$ is weakly δ-continuous; Y is rim-compact and $G(f)$ is δ-closed with respect to X, then f is super continuous.*

Proof. Clearly from Theorem 3.1 $f : (X, \tau_s) \to (Y, \nu)$ is weakly continuous and from Theorem 3.3. $G(f)$ is closed in $(X \times Y, \tau_s \times \nu)$. From Theorem 3 of Noiri [7], $f: (X, \tau_s) \to (Y, \nu)$ is continuous. Hence $f: (X, \tau) \to (Y, \nu)$ is super continuous.

In the same way Theorem 9 of Baker [2] is a direct consequence of Corollary 1 of Noiri [7] and Theorem 3.3.

THEOREM 3.7. *If $f : (X, \tau) \to (Y, \nu)$ is faintly continuous and (Y, ν) is almost regular (regular) then f is almost strongly θ-continuous (strongly θ-continuous).*

Proof. Let $f : (X, \tau) \to (Y, \nu)$ be faintly continuous and (Y, ν) is almost regular. Since $f : (X, \tau) \to (Y, \nu_\theta)$ is continuous and $\nu_\theta = \nu_s$ we have $f : (X, \tau) \to (Y, \nu_s)$ is continuous. We have from Theorem 1 of Mrsevic, Reilly, Vamanamurty [6] that (Y, ν_s) is regular. Hence f is a continuous function to a regular space. Now from Theorem 8 of Long and Herrington [3] $f : (X, \tau) \to (Y, \nu_s)$ is strongly θ-continuous. So $f : (X, \tau_\theta) \to (Y, \nu_s)$ is continuous equivalently $f : (X, \tau) \to (Y, \nu)$ is almost strongly θ-continuous.

The other result can be proved in a similar way.

COROLLARY 3.8. *If the range space is almost regular (regular) all types of functions between almost strong θ-continuity (strong θ-continuity) and faint continuity are equivalent.*

Theorem 4.1 and Corollary 4.3 of Noiri, Kang [9] are direct consequences of Theorem 3.7.

The following Corollary is a generalization of Corollary of Theorem 9 of Baker [2].

COROLLARY 3.9. *If $f : (X, \tau) \to (Y, \nu)$ is faintly continuous and Y is Hausdorff and rim-compact then f is strongly θ-continuous.*

Proof. Clear from theorem 4 of Noiri [7] that Hamadorff and rim-compact space is regular.

Theorem 10 of Baker [2] is a direct consequence of Theorem 3.3 and the fact that a function which has closed graph and compact range is continuous.

The following Corollary is a generalization of Theorem 5 of Baker [2].

COROLLARY 3.10. *If $f, g : X \to Y$ are δ-continuous functions and Y is Hausdorff then $A = \{x \in X : f(x) = g(x)\}$ is δ-closed.*

Proof. Clearly $f : (X, \tau_s) \to (Y, \nu_s)$ *is continuous, and* (Y, ν_s) is Hausdorff from Proposition 1 of Mrsevic', Reilly and Vamanamurty [6]. Hence A is δ-closed.

4. Pre-open sets and weak continuity. A set A is called pre-open (semi-open) if $A \subset \mathring{\bar{A}} (A \subset \bar{\mathring{A}})$ (see Allam, Zahran, Hasanein'). Pre-open sets are called almost open in the paper of Rose and Jankovic', [14].

A function $f : X \to Y$ is called almost open [16] if for every open set V in Y $f^{-1}(\bar{V}) \subset \overline{f^{-1}(V)}$ and pre-open [5] if for every open set U in X $f(U)$ is pre-open. Rose [13] showed that almost openness is equivalent to pre-openness.

THEOREM 4.1. *$f : X \to Y$ us weakly continuous iff for each pre-open set $A \subset Y$, $\overline{f^{-1}(A)} \subset f^{-1}(\bar{A})$.*

Proof. Let $f : X \to Y$ be weakly continuous and A be a pre-open set. Then $A \subset \mathring{\bar{A}} \subset \bar{\mathring{\bar{A}}} \subset \bar{A}$, $f^{-1}(A) \subset f^{-1}(\mathring{\bar{A}}) \subset f^{-1}(\bar{\mathring{\bar{A}}}) \subset f^{-1}(\bar{A})$. Since f is weakly continuous and $\mathring{\bar{A}}$ is an open set we get $f^{-1}(\bar{\mathring{\bar{A}}}) \subset f^{-1}(\bar{\mathring{\bar{A}}}) \subset f^{-1}(\mathring{\bar{A}})$ from the fact thet $f : X \to Y$ is weakly continuous iff for each open set V in Y, $\overline{f^{-1}(V)} \subset f^{-1}(\bar{V})$ Rose and Janković [14]. Hence $\overline{f^{-1}(A)} \subset f^{-1}(\bar{\mathring{\bar{A}}}) \subset f^{-1}(\bar{A})$.

The converse is clear from the property mentioned above.

Theorem 2.2. and Theorem 2.3 of Allam, Zahran and Hasanein [1] is a corollary of Theorem 3.1 which generalizes Lemma 2.14 of Allam, Zahran and Hasanein [1].

COROLLARY 4.2. (Theorem 2.3 of Allam, Zahran, Hasanein [1]). *Let $f : X \to Y$ be an almost continuous mapping. Then for each pre-open set $A \subset Y$, $\overline{f^{-1}(A)} \subset f^{-1}(\bar{A})$.*

The following Lemma is a generalization of Lemma 2.13 of Allam, Zahran and Hasanein [1] and contains a short proof.

LEMMA 4.3. *$f : X \to Y$ is weakly continuous iff for every pre-open set A in Y, $f^{-1}(A) \subset intf^{-1}(\bar{A})$.*

Proof. Let f be weakly continuous and A be a pre-open set in Y. We know that $f : X \to Y$ is weakly continuous iff for each open set V in Y $f^{-1}(V) \subset \mathrm{int}(f^{-1}(\bar{V}))$. Since $A \subset \mathring{\bar{A}}$ we have $f^{-1}(A) \subset f^{-1}(\mathring{\bar{V}}) \subset \mathrm{int}(f^{-1}(\bar{\mathring{\bar{A}}})) \subset \subset \mathrm{int} f^{-1}(\bar{A})$

The converse is clear.

LEMMA 4.4. *If $f : X \to Y$ is pre-open function then for every semi-open set B in Y, $f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$.*

Proof. Let $f: X \to Y$ be pre-open and B be a semi-open set. Clearly $B \subset \overline{\mathring{B}}$ and $\bar{B} = \overline{\mathring{B}}$.

$f^{-1}(\bar{B}) = f^{-1}(\overline{\mathring{B}}) \subset \overline{f^{-1}(\mathring{B})} \subset \overline{f^{-1}(\bar{B})}$ since f is pre-open and $\mathring{B}$ is open.

LEMMA 4.5. *$f: X \to Y$ u is pre-open and almost continuous iff for every semi-open set B in Y $f^{-1}(\bar{B}) = \overline{f^{-1}(\bar{B})}$.*

Proof. It is clear from Lemma 4.4. the definition of almost openness which is equivalent to pre-openness, the Theorem of Mamata Deb (it is seen from Corollary 2 of Prakash and Srivastava [10]) and the fact that every open set is semi-open.

COROLLARY 4.6. *$f: X \to Y$ is weakly continuous and pre-open iff for each open set V in Y $\overline{f^{-1}(V)} = f^{-1}(\bar{V})$.*

Proof. It is clear from the fact that f is pre-open (resp. weakly continuous) iff for each open set V in Y $f^{-1}(\bar{V}) \subset \overline{f^{-1}(V)}$ (resp. $\overline{f^{-1}(V)} \subset f^{-1}(\bar{V})$).

Theorem 2.7 of Allam, Zahran, Hasanein [1] is a direct consequence the following corollary which generalizes Corollary 2.5 of Allam, Zahran, Hasanein [1].

COROLLARY 4.7. *$f: X \to Y$ is almost] continuous and pre-open iff for each open set V in Y, $\overline{f^{-1}(V)} = f^{-1}(\bar{V})$.*

Proof. Clearly the pre-open and weakly continuous function is almost continuous from Theorem 2.2 of Singal and Singal [15] and Lemma 4.4 of Noiri [8] and the almost continuous function is weakly continuous. Proof is clear now from Corollary 4.6.

LEMMA 4.8. *If A is a pre-open subset of X then τ_s/A (i.e. induced topology on X by τ_s) is a subset of $(\tau/A)_s$.*

Proof. If U is regular open set in (X, τ) then $A \cap U$ is regular open in the subspace τ/A from Lemma 2.8 of Allam, Zahran, Hasanein [1]. Hence $\tau_s/A \subset (\tau/A)_s$.

The fllowing theorem is given by Allan, Zahran, Hasanein [1], but we will give a different proof of it.

THEOREM 4.9. (Theorem 2.9. of Allan, Zahran, Hassanein [1]). *If $f: (X, \tau) \to (Y, \nu)$ is δ-continuous and A is pre-open in (X, τ), then $f/A: (A, \tau_A) \to (Y, \nu)$ is δ-continuous.*

Proof. Since f if δ-continuous, $f: (X, \tau_s) \to (Y, \nu_s)$ is continuous. Clearly $f/A: (A, \tau_s/A) \to (Y, \nu_s)$ is continuous and from Lemma 4.8 $f/A: (A, (\tau/A)_s) \to (Y, \nu_s)$ is continuous. Hence $f/A: (A, \tau/A) \to (Y, \nu)$ is δ-continuous.

THEOREM 4.10. *If $f: (X, \tau) \to (Y, \nu)$ is super continuous (super faintly continuous) and A is pre-open set in X then $f/A: (A, \tau/A) \to (Y, \nu)$ is super continuous (super faintly continuous).*

Proof. It is similar to the proof of Theorem 4.9. Theorem 6 of Baker [2] is a corollary of Theorem 4.10.

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