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ON INEQUALITIES FOR INDEFINITE FORM

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1. In this paper are given proofs of the Aczél, Popoviciu and Bellman inequalities concerning an indefinite form, by the method of a common fixed point [3]. Also, some premises are corrected and added the necessary and sufficient conditions when in the Popoviciu and Bellman inequalities hold equalities.

The indefinite form is [χ]²

$$\Phi(x) = (x_1^p - x_2^p - \ldots - x_n^p)^{1/p}, \quad p \geqslant 1$$

for values of the x_i in the region R defined by

(a) $x_i \geqslant 0$
(b) $x_1 > (x_2^p + x_3^p + \ldots + x_n^p)^{1/p}$.

The next theorem due to J. Aczél, 1956. [4].

THEOREM A. *Let* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *be two sequences of real numbers, such that*

$$a_1^2 - a_2^2 - \ldots - a_n^2 > 0, \text{ or } b_1^2 - b_2^2 - \ldots - b_n^2 > 0.$$

Then

$$(a_1^2 - a_2^2 - \ldots - a_n^2)(b_1^2 - b_2^2 - \ldots - b_n^2) \leqslant (a_1b_1 - a_2b_2 - \ldots - a_nb_n)^2,$$

with equality if and only if the sequences a and b are proportional.

2. The Aczél inequality was generalised by T. Popoviciu

$$(1)\quad (a_1^p - a_2^p - \ldots - a_n^p)(b_1^p - b_2^p - \ldots - b_n^p) \leqslant (a_1b_1 - a_2b_2 - \ldots - a_nb_n)^p.$$

The conditions

$$(2)\qquad a_1^p - a_2^p - \ldots - a_n^p > 0, \text{ or } b_1^p - b_2^p - \ldots - b_n^p > 0$$

and $p \geqslant 1$ given in [4] are not sufficient. The counterexample is $p = 3$, $a = b = (2, 1, 1, 1)$ when (1) becomes $5 \cdot 5 \leqslant 1$. For $p > 2$, $n = 2$ and $a_1 > a_2$ the converse inequality holds

$$(3)\quad (a_1^p - a_2^p)^2 > (a_1^2 - a_2^2)^p \text{ or } t - 1 > (t^q - 1)^{\frac{1}{q}},\ t = \left(\frac{a_1}{a_2}\right)^p,\ q = \frac{2}{p}.$$

The function $y(t) = (t^q - 1)^{1/q}$, $t \geqslant 1$, $0 < q < 1$ has a derivate $y'(t) = (1 - t^{-q})^{\frac{1-q}{q}}$ with boundaries $0 \leqslant y'(t) < 1$, so we obtain (3).

THEOREM B. *If* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are sequences of nonnegative real numbers such that*

$$(4) \qquad a_1^p - a_2^p - \ldots - a_n^p \geqslant 0 \ \text{and} \ b_1^p - b_2^p - \ldots - b_n^p \geqslant 0,$$

then, for $0 < p \leqslant 2$,

$$(5) \quad (a_1^p - a_2^p - \ldots - a_n^p)^{1/p} (b_1^p - b_2^p - \ldots - b_n^p)^{1/p} \leqslant a_1 b_1 - a_2 b_2 - \ldots - a_n b_n,$$

and conversely for $p < 0$.

If $p < 2$ *equality holds if and only if* $a = (a_1, 0, \ldots, 0)$ *and* $b = (b_1 0, \ldots, 0)$. *If* $p = 2$ *equality holds if and only if* a *and* b *are proportional.*

Proof. Let $0 < p \leqslant 2$,

$$X = \{(x_1, \ldots, x_n) \mid x_1 \geqslant 0, \ldots, x_n \geqslant 0,$$
$$x_1^p - x_2^p - \ldots - x_n^p = a_1^p - a_2^p - \ldots - a_n^p\},$$
$$Y = \{(y_1, \ldots, y_n) \mid y_1 \geqslant 0, \ldots, y_n \geqslant 0,$$
$$y_1^p - y_2^p - \ldots - y_n^p = b_1^p - b_2^p - \ldots - b_n^p\},$$

and a functional $f : X \times Y \to R$

$$f(x, y) = (x_1 y_1 - x_2 y_2 - \ldots - x_n y_n)^p - (x_1^p - x_2^p - \ldots - x_n^p)(y_1^p - y_2^p - \ldots - y_n^p).$$

If $a_1 = a_i$ or $b_1 = b_i$ for some $i \in \{2, \ldots, n\}$, then (5) is trivial. Hence, suppose that $a_1 > a_i$ and $b_1 > b_i$ for all $i = 2, \ldots, n$.

Let $\alpha = \dfrac{a_i}{a_1}$, $\beta = \dfrac{b_i}{b_1}$ and $\alpha < \beta$. Let

$$(6) \qquad b' = (b_1', b_2, \ldots, b_{i-1}, b_i', b_{i+1}, \ldots, b_n),$$

where b_1' and b_i' are such that

$$(7) \qquad b_1'^p - b_i'^p = b_1^p - b_i^p, \quad b_1' : b_i' = a_1 : a_i.$$

Conditions (7) are satisfied if

$$b_1' = \delta a_1, \quad b_i' = \delta a_i, \quad \delta = \left(\frac{b_1^p - b_i^p}{a_1^p - a_i^p} \right)^{\frac{1}{p}}.$$

Inequality $\alpha < \beta$ implies that $b_1' < b_1$ and $b_i' < b_i$.

Let us now demonstrate that $f(a, b) > f(a, b')$ i.e.

$$a_1 b_1 - a_i b_i > a_1 b_1' - a_i b_i' = \delta(a_1^2 - a_i^2),$$

or

$$(8) \qquad \frac{b_1^p - b_i^p}{a_1^p - a_i^p} < \left(\frac{a_1 b_1 - a_i b_i}{a_1^2 - a_i^2} \right)^p, \quad \frac{1 - \beta^p}{(1 - \alpha\beta)^p} < \frac{1 - \alpha^p}{(1 - \alpha\alpha)^p}.$$

Let $g(t) = \dfrac{1 - t^p}{(1 - \alpha t)^p}$, $\alpha \leqslant t \leqslant 1$, so that $g'(t) = \dfrac{p(\alpha - t^{p-1})}{(1 - \alpha t)^{p+1}}$.

If $0 < p \leqslant 1$ then $\alpha < 1 \leqslant t^{p-1}$, if $1 < p \leqslant 2$ then $\alpha \leqslant \alpha^{p-1} \leqslant t^{p-1}$, so that $g'(t) < 0$, $\alpha < t < 1$, what implies (8).

In the case that $\alpha > \beta$, similarly defines a'.

Mappings $F_i : X \times Y \to X \times Y$, $i = 2, \ldots, n$

$$F_i(x, y) = \begin{cases} (x, y'), & \text{if } x_i y_1 \leqslant x_1 y_i \\ (x', y), & \text{if } x_i y_1 \geqslant x_1 y_i \end{cases},$$

are in accordance with the functional

$$f(x, y) \geqslant f(F_i(x, y)).$$

Let

$$(9) \qquad F_{i_1}, F_{i_2}, \ldots, F_{i_m}, \ldots$$

be a sequence of mappings $F_2, \ldots, F_n$ in which each of these mappings infinitely times appears. Application of (9) generates a sequence of vectors

$$(10) \qquad (a, b), (a^{(1)}, b^{(1)}) = F_{i_1}(a, b), \ldots$$
$$\ldots, (a^{(m)}, b^{(m)}) = F_{i_m}(a^{(m-1)}, b^{(m-1)}), \ldots.$$

The coordinates in (10) nonincrease so that

$$\lim_{m \to \infty} (a^{(m)}, b^{(m)}) = (c, d) \in X \times Y$$

and

$$(11) \qquad f(a, b) \geqslant f(a^{(1)}, b^{(1)}) \geqslant \ldots \geqslant f(a^{(m)}, b^{(m)}) \geqslant \ldots \geqslant f(c, d).$$

The mapping F_k, $k \in \{2, \ldots, n\}$ is continuous so that

$$\lim_{m \to \infty} F_k(a^{(m)}, b^{(m)}) = F_k(c, d).$$

Let $\{j_m\}$ be a sequence of indexes such that $i_{j_m} = k$, $m \in N$. The sequence

$$(a^{(i_{j_m}+1)}, b^{(i_{j_m}+1)}) = F_k(a^{(i_{j_m})}, b^{(i_{j_m})}), \quad m = 1, 2, \ldots$$

is a subsequence of both convergent sequences

$$\{(a^{(m)}, b^{(m)})\} \text{ and } \{F_k(a^{(m)}, b^{(m)})\}, \text{ so they}$$

converge to the same limit $(c, d) = F_k(c, d)$. Hence, (c, d) is a common fixed point for all mappings $F_2, \ldots, F_n$, what implies proportionality $d = rc$, $r > 0$.

It remains to prove that

$$f(c, c) = (c_1^2 - c_2^2 - \ldots - c_n^2)^p - (c_1^p - c_2^p - \ldots - c_n^p)^2 \geqslant 0.$$

Let $i \in \{2, \ldots, n\}$,

$$h_i(t) = (t^2 - c_2^2 - \dots - c_{i-1}^2 - \bar{t}^2 - \dots - c_n^2)^p -$$
$$- (t^p - c_2^p - \dots - c_{i-1}^p - \bar{t}^p - \dots - c_n^p)^2,$$
$$t \geqslant \sqrt{c_1^2 - c_i^2},\ t^2 - \bar{t}^2 = c_1^2 - c_i^2, \text{ and so } t - \bar{t}\bar{t}' = 0.$$

A derivate is

$$h_i'(t) = -2(t^p - c_2^p - \dots - \bar{t}^p - \dots - c_n^p)\, pt\, (t^{p-2} - \bar{t}^{p-2}) \geqslant 0,$$

so that

$$f(c,\ c) \geqslant f(H_i(c),\ H_i(c)),$$

where

$$H_i(c) = (\sqrt{c_1^2 - c_i^2},\ c_2,\ \dots,\ c_{i-1},\ 0,\ c_{i+1},\ \dots,\ c_n).$$

Therefore

$$f(c,\ c) \geqslant f(H_2(c),\ H_2(c)) \geqslant \dots \geqslant$$
$$\geqslant f(H_n(\dots H_2(c) \dots),\ H_n(\dots H_2(c) \dots)) = \tag{12}$$
$$= f((\sqrt{c_1^2 - c_2^2 - \dots - c_n^2},\ 0, \dots, 0),\ (\sqrt{c_1^2 - c_2^2 - \dots - c_n^2},\ 0, \dots, 0)) = 0.$$

If $p < 0$ then

$$\Phi(a)\Phi(b) \geqslant a_1 b_1 \geqslant a_1 b_1 - a_2 b_2 - \dots - a_n b_n,$$

and it is obvious when equality holds. If $p = 0$ conditions (4) are not satisfied $(n > 2)$, so that theorem is valid.

For $p < 2$ in (5) equality holds if and only if it holds in sequences (11) and (12), what implies $a = (a_1,\ 0, \dots, 0)$ and $b = (b_1,\ 0, \dots, 0)$. If $p = 2$ then $f(c,\ c) = 0$, so that equality holds only for proportional sequences.

3. For the R. Bellman inequality (13) in [4] it is supposed same condition (2). The counterexample is $p = 3$, $a = (\sqrt[3]{9}, 2, 0, \dots, 0)$, $b = (0, 1, 0, \dots, 0)$, when (13) becomes $1 + (-1) \leqslant \sqrt[3]{-18}$. In the original paper [2] and also in [1], the premise is sharper

$$a_1^p - a_2^p - \dots - a_n^p > 0 \text{ and } b_1^p - b_2^p - \dots - b_n^p > 0,$$

what is weaken in the next theorem.

THEOREM C. *If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are sequences of nonnegative real numbers which satisfy*

$$a_1^p - a_2^p - \dots - a_n^p \geqslant 0 \text{ and } b_1^p - b_2^p - \dots - b_n^p \geqslant 0,$$

then, for $p > 1$

$$(a_1^p - a_2^p - \dots - a_n^p)^{1/p} + (b_1^p - b_2^p - \dots - b_n^p)^{1/p} \leqslant \tag{13}$$
$$\leqslant ((a_1 + b_1)^p - (a_2 + b_2)^p - \dots - (a_n + b_n)^p)^{1/p}.$$

Equality holds if and only if a and b are proportional.

Proof. Let

$$f(a,\ b) = \Phi(a,\ b) - \Phi(a) - \Phi(b). \tag{14}$$

Inequality $f(a,\ b) \geqslant f(a,\ b')$, v. (6), is equivalent to

$$(a_1 + b_1)^p - (a_i + b_i)^p \geqslant (a_1 + \delta a\,)^p - (a_i + \delta a_i)^p =$$
$$= (a_1^p - a_i^p)(1 + \delta)^p =$$
$$= ((a_1^p - a_i^p)^{1/p} + (b_1^p - b_i^p)^{1/p})^p$$

or

$$(a_1^p - a_i^p)^{1/p} + (b_1^p - b_i^p)^{1/p} \leqslant ((a_1 + b_1)^p - (a_i + b_i)^p)^{1/p}, \tag{15}$$

what is inequality (13) for $n = 2$.

For the sake of determination, let $\dfrac{a_i}{a_1} \leqslant \dfrac{b_i}{b_1}$.

The function

$$g(t) = ((t + b_1)^p - (a_i + b_i)^p)^{1/p} - (t^p - a_i^p)^{1/p} - (b_1^p - b_i^p)^{1/p},$$
$$t \geqslant \frac{b_1}{b_i} a_i \text{ has a derivate}$$
$$g'(t) = ((t + b_1)^p - (a_i + b_i)^p)^{\frac{1-p}{p}} (t + b_1)^{p-1} - (t^p - a_i^p)^{\frac{1-p}{p}} t^{p-1}.$$

The relation $g'(t) \geqslant 0$ can be reduced to

$$(t + b_1)^p (t^p - a_i^p) \geqslant ((t + b_1)^p - (a_i + b_i)^p) t^p,$$
$$\text{i.e. } t^p (a_i + b_i)^p \geqslant a_i^p (t + b_1)^p \text{ and } tb_i \geqslant b_1 a_i.$$

Hence, $g(a_1) \geqslant g\left(\dfrac{b_1}{b_i} a_i\right) = 0$, what proves (15).

Like in the proof of the foregoing theorem, it can be formed a sequence (11). The function (14) has nought value for proportional sequences, **c and d, when only equality-sign holds.**

In this proof it can be transformed only one vector, e.g. b, $b^{(1)}, \dots$ $\dots, b^{(m)}, \dots$, **which converges to a vector** proportional to a.

REFERENCES

1. Beckenbach E. F., Bellman R., *Inequalities*, Berlin—Heidelberg—New York, (1961), 36—39.
2. Bellman R., *On an inequality concerning an indefinite form*, Amer. Math. Montly **63** (1956), 108—109.
3. Bjelica M., *Fixed point and inequalities* (Serbocroatian), doctorial thesis, Belgrade, 1990.
4. Mitrinović D. S., Vasić P. M., *Analitic inequalities*, Berlin—Heidelberg—New York, 1970, 57—59.
5. Popoviciu T., *On an inequality* (Romanian), Gaz. Mat. Fiz. A **11** (64) (1959), 451—461.

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