MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 19, N° 2, 1990, pp. 111–121

SPLINE APPROXIMATION WITH PRESERVATION OF MOMENTS

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0. Introduction. The problem of approximating an integrable function f on the interval $[0,1]$ by a spline function is considered. The spline function is a polynominal spline with fixed degree and variable knots. The multiplicities of the knots are specified. The spline approximation is constructed in such a way as to preserve as many initial moments of f as possible. For the simple knots of the spline function the problem is solved in [3].

We consider the two methods presented in [3] to solve our problem, namely first when the solution of the problem is obtained by certain moment functionals and second when the solution is related to construct some generalized Gauss-Lobatto quadrature formula. The conditions which assure the existence and uniqueness of the spline approximation are presented. The error of the approximation is obtained by the remainder term from generalized Gauss-Lobatto quadrature or corresponding generalized Gauss-Christoffel quadrature formula. Finally, some numerical examples are presented.

1. Statement of the problem. Let $s_{r,m}$ be a polynomial spline of degree m with the knots t_ν, $\nu=\overline{1,n}$, $0<t_1<\ldots<t_n<1$ and the corresponding multiplicities given by $r=(r_1,\ldots,r_n)$, $r_\nu\leqslant m+1$, $\nu=\overline{1,n}$. The spline function $s_{r,m}$ can be written in the form

$$s_{r,m}(t)=p_m(t)+\sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1}a_{\nu\mu}(t_\nu-t)_+^{m-\mu} \tag{1.1}$$

where $p_m(t)$ is a polynomial of degree m.

This form is equivalent with the following

$$s_{r,m}(t)=p_m(t)+\sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1}\alpha_{\nu\mu}\left[\frac{d^\mu}{dx^\mu}(x-t)_+^m\right]_{x=t_\nu} \tag{1.2}$$

where $a_{\nu,\mu}=m^{[\mu]}\alpha_{\nu\mu}$, $m^{[\mu]}=m(m-1)\ldots(m-\mu+1)$. Also, for the polynomial $p_m(t)$ one considers the representation

$$p_m(t)=\sum_{i=0}^{m}\beta_i m^{[i]}(1-t)^i,$$

with

$$\beta_i = \frac{(-1)^i}{m!} p_m^{(i)}(1), \quad i = \overline{0, m}.$$

Certainly, a spline function in the form (1.1) or (1.2) is completely determinated when the coefficients $a_{\nu\mu}$ or $\alpha_{\nu\mu}$, $p_m(t)$ or β_i, and the knots t_ν, $\nu = \overline{1, n}$, are known.

The problem is to construct a spline function $s_{r,m}(t)$ so that

$$\int_0^1 t^j s_{r,m}(t)\mathrm{d}t = \int_0^1 t^j f(t)\mathrm{d}t, \quad j = \overline{0, N+n+m}, \tag{1.3}$$

where f is an integrable function on the interval $[0, 1]$, and $N = r_1 + \ldots + r_n$. In other words, the spline function $s_{r,m}(t)$ must reproduce all the moments of the function f of order less or equal to $N + n + m$.

2. Solution by moment functionals. Taking into account that

$$t^j = \frac{1}{(m+j+1)^{[m+1]}} \frac{\mathrm{d}^{m+1}}{\mathrm{d}t^{m+1}} (t^{m+j+1})$$

and using the generalized formula of the integration by parts we have

$$\int_0^1 t^j p_m(t)\mathrm{d}t = \frac{j!m!}{(m+j+1)!} \sum_{i=0}^{m} \beta_i \left[\frac{\mathrm{d}^{m-i}}{\mathrm{d}t^{m-i}} (t^{m+1+j})\right]_{t=1}$$

with the above specified β_i. On the other hand, using the Euler function of the first kind it results that

$$\int_0^1 (t_\nu - t)_+^{m-\mu} t^j \mathrm{d}t = \int_0^{t_\nu} (t_\nu - t)^{m-\mu} t^j \mathrm{d}t = \frac{j!(m-\mu)!}{(m-\mu+j+1)!} t_\nu^{m-\mu+j+1}.$$

On the basis of these relations we have that (1.3) is equivalent to

$$\frac{j!m!}{(m+j+1)!} \sum_{i=0}^{m} \beta_i \left[\frac{\mathrm{d}^{m-i}}{\mathrm{d}t^{m-i}} (t^{m+1+j})\right]_{t=1} +$$

$$+ \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu - 1} \frac{j!(m-\mu)!}{(m-\mu+j+1)!} a_{\nu\mu} t_\nu^{m-\mu+j+1} = \int_0^1 t^j f(t)\mathrm{d}t,$$

$$j = \overline{0, N+n+m},$$

and also

$$\sum_{i=0}^{m} \beta_i \left[\frac{\mathrm{d}^{m-i}}{\mathrm{d}t^{m-i}} (t^{m+1+j})\right]_{t=1} + \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu - 1} \alpha_{\nu\mu} \left[\frac{\mathrm{d}^{\mu}}{\mathrm{d}t^{\mu}} (t^{m+j+1})\right]_{t=t_\nu} =$$

$$= \frac{(m+j+1)!}{j!m!} \int_0^1 t^j f(t)\mathrm{d}t, \quad j = \overline{0, N+n+m}. \tag{2.1}$$

One denotes

$$\delta_j = \delta_j(f) = \frac{(m+j+1)!}{j!m!} \int_0^1 t^j f(t)\mathrm{d}t,$$

and then one defines a linear functional $\mathscr{L}$ on the set of polynomials of the form $t^{m+1} \cdot p(t)$, $p \in \mathscr{P}_{N+n+m}$, by

$$\mathscr{L}(t^{m+1} \cdot t^j) = \delta_j, \quad j = \overline{0, N+n+m}.$$

Using this linear functional one defines the inner product for any polynomials p and q with $p \cdot q \in \mathscr{P}_{N+n-1}$ in the following manner

$$(p, q) = \mathscr{L}(t^{m+1}(1-t)^{m+1} p(t) q(t)). \tag{2.2}$$

One considers (if it exists) the monic polynomial $\omega_N = \omega_N(\cdot; \mathscr{L})$ of degree N orthogonal with respect to the inner product (2.2) to all polynomials of lower degree than n, i.e.

$$\omega_N(t) = t^N + \cdots \text{ and } (\omega_N, q) = 0, \text{ for all } q \in \mathscr{P}_{n-1}.$$

THEOREM 2.1. *There exists a unique spline function of the form (1.1) or (1.2), which satisfies the moment relations (1.3) if and only if the orthogonal polynomial* $\omega_N = \omega_N(\cdot; \mathscr{L})$ *exists uniquely and it has n real zeros* $t_\nu^{(N)}$, $\nu = \overline{1, n}$, *with the multiplicities* r_ν, $\nu = \overline{1, n}$, *and* $0 < t_1^{(N)} < \ldots < t_n^{(N)} < 1$. *Moreover, in this case we have that* $t_\nu = t_\nu^{(N)}$, $\nu = \overline{1, n}$, *and the coefficients* β_i *and* $\alpha_{\nu\mu}$ *are uniquely determinated from the condition*

$$\mathscr{L}_0(t^{m+1} \cdot p(t)) = \mathscr{L}(t^{m+1} \cdot p(t)), \quad p \in \mathscr{P}_{N+m},$$

where

$$\mathscr{L}_0(g) = \sum_{i=0}^{m} \beta_i g^{(m-i)}(1) + \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu - 1} \alpha_{\nu\mu} g^{(\mu)}(t_\nu). \tag{2.3}$$

Proof. Using (2.3), equations (2.1) can be written in the form

$$\mathscr{L}_0(t^{m+1} \cdot t^j) = \delta_j, \quad j = \overline{0, N+n+m},$$

or equivalently

$$\mathscr{L}_0(t^{m+1} \cdot p(t)) = \mathscr{L}(t^{m+1} \cdot p(t)), \quad p \in \mathscr{P}_{N+n+m}. \tag{2.4}$$

From the existence and the uniqueness of the spline approximation (1.2) which satisfies conditions (1.3) we have that the linear functional $\mathscr{L}_0$ is well defined and (1.3) is equivalent with (2.4). Using the knots of the spline functions (1.2) one considers the polynomial

$$\omega_N(t) = \prod_{\nu=1}^{n} (t - t_\nu)^{r_\nu},$$

and by (2.4)

$$(\omega_N, q) = \mathscr{L}(t^{m+1}(1-t)^{m+1}\omega_N(t)q(t)) = \mathscr{L}_0(t^{m+1}(1-t)^{m+1}\omega_N(t)q(t)),$$

for any polynomial $q \in \mathscr{P}_{n-1}$. Taking into account the definition of the functional $\mathscr{L}_0$ it results that $(\omega_N, q) = 0$. This proves the necessity of the condition.

To prove the sufficiency we consider a polynomial $p \in \mathscr{P}_{N+n+m}$. Then we have $p(t) = (1-t)^{m+1}\omega_N(t)q(t) + r(t)$, with $q \in \mathscr{P}_{n-1}$, $r \in \mathscr{P}_{N+m}$ and ω_N the **monic polynomial of degree** N **orthogonal with respect** to inner product (2.2) **to all polynomials of degree less** than n, **and which** has the real zeros $t_\nu^{(N)}$, $\nu = \overline{1, n}$, $0 < t_1^{(N)} < \ldots < t_n^{(N)} < 1$, with the multiplicities r_ν, $\nu = \overline{1, n}$.

One writes succesively

$$\mathscr{L}(t^{m+1} \cdot p(t)) = \mathscr{L}(t^{m+1}(1-t)^{m+1}\omega_N(t)q(t)) + \mathscr{L}(t^{m+1} \cdot r(t)) = \mathscr{L}(t^{m+1} \cdot r(t)).$$

On the other hand one considers $\mathscr{L}_0$ defined by (2.3) with $t_\nu = t_\nu^{(N)}$, $\nu = \overline{1, n}$, and β_i, $i = \overline{0, m}$, $\alpha_{\nu\mu}$, $\nu = \overline{1, n}$, $\mu = \overline{0, r_\nu - 1}$ unknowns, then

$$\mathscr{L}_0(t^{m+1} \cdot p(t)) = \mathscr{L}_0(t^{m+1}(1-t)^{m+1}\omega_N(t)q(t)) + \\ + \mathscr{L}_0(t^{m+1} \cdot r(t)) = \mathscr{L}_0(t^{m+1} \cdot r(t)).$$

The coefficients β_i and $\alpha_{\nu\mu}$ are uniquely determined as solution of the linear system

$$\mathscr{L}_0(t^{m+1} \cdot t^j) = \mathscr{L}(t^{m+1} \cdot t^j), \quad j = \overline{0, N+m},$$

which has a generalized Vandermonde determinant.

3. Solution by generalized Gauss-Lobatto quadrature. In the following our problem is related to the problem of constructing a certain generalized **Gauss-Lobatto quadrature** and of the corresponding generalized **Gauss-Christoffel quadrature formula. One** considers that $f \in C^{m+1}[0,1]$ and the values $f^{(k)}(1)$, $k = \overline{0, m}$, are known.

If the generalized integration by parts formula is applied to integral from the right side in (2.1) it results that

$$\sum_{i=0}^{m} \beta_i \left[\frac{\mathrm{d}^{m-i}}{\mathrm{d}t^{m-i}} (t^{m+1+j}) \right]_{t=1} + \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu - 1} \alpha_{\nu\mu} \left[\frac{\mathrm{d}^\mu}{\mathrm{d}t^\mu} (t^{m+1+j}) \right]_{t=t_\nu} =$$

$$\text{(3.1)} \qquad = \sum_{i=0}^{m} \gamma_i \left[\frac{\mathrm{d}^{m-i}}{\mathrm{d}t^{m-i}} (t^{m+1+j}) \right]_{t=1} + \frac{(-1)^{m+1}}{m!} \int_0^1 f^{(m+1)}(t) t^{m+1+j} \mathrm{d}t,$$

$$j = \overline{0, N+n+m}, \quad \gamma_i = \frac{(-1)^i}{m!} f^{(i)}(1), \quad i = \overline{0, m}.$$

If one defines the measure

$$\mathrm{d}\lambda_m(t) = \frac{(-1)^{m+1}}{m!} f^{(m+1)}(t) \mathrm{d}t$$

on $[0,1]$, then the relations (3.1) are equivalent to

$$\text{(3.2)} \qquad \mathscr{L}_0(t^{m+1} \cdot p(t)) = \hat{\mathscr{L}}(t^{m+1} \cdot p(t)),$$

where $p \in \mathscr{P}_{N+n+m}$, and $\mathscr{L}_0$ is defined as in the previous section and

$$\hat{\mathscr{L}}(g) = \sum_{i=0}^{m} \gamma_i g^{(m-i)}(1) + \int_0^1 g(t) \mathrm{d}\lambda_m(t).$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product defined by means of the functional $\hat{\mathscr{L}}$:

$$\langle p, q \rangle = \hat{\mathscr{L}}(t^{m+1}(1-t)^{m+1} p(t)q(t)) = \\ = \int_0^1 t^{m+1}(1-t)^{m+1} p(t)q(t) \mathrm{d}\lambda_m(t).$$

One considers (if it exists) the monic polynomial $\hat{\omega}_N = \omega_N(\cdot\,; \hat{\mathscr{L}})$ of degree N **orthogonal** with respect to inner product $\langle \cdot, \cdot \rangle$ to all polynomials of lower degree than n, i.e. $\hat{\omega}_N(t) = t^N + \cdots$, and $\langle \hat{\omega}_N, q \rangle = 0$, for all $q \in \mathscr{P}_{n-1}$.

An analogous theorem with theorem from the previous section holds.

THEOREM 3.1. *If* $f \in C^{m+1}$ $[0,1]$, *then there exists a unique spline function of the form (1.1) or (1.2), which satisfies the moment equations (1.3) if and only if the orthogonal polynomial* $\hat{\omega}_N = \hat{\omega}_N(\cdot\,; \hat{\mathscr{L}})$ *uniquely exists and it has* n *real zeros* $\hat{t}_\nu^{(N)}$, $\nu = \overline{1, n}$, *with the multiplicities*, r_ν, $\nu = \overline{1, n}$, *and* $0 < \hat{t}_1^{(N)} < \ldots < \hat{t}_n^{(N)} < 1$. *Moreover, in this case we have that* $t_\nu = \hat{t}_\nu^{(N)}$, $\nu = \overline{1, n}$, *and the coefficients* β_i *and* $\alpha_{\nu\mu}$ *are obtained from the condition*

$$\mathscr{L}_0(t^{m+1} \cdot p(t)) = \hat{\mathscr{L}}^{m+1}(t^{m+1} \cdot p(t)), \; p \in \mathscr{P}_{N+m}.$$

The proof follows in the same a way as the proof of theorem 2.1.

Using the results of this theorem one relates the solution of our problem to construction of some generalized Gauss-Lobatto quadrature or **corresponding generalized Gauss-Christoffel quadrature.**

We consider generalized Gauss-Lobatto quadrature formula

$$\text{(3.3)} \qquad \int_0^1 g(t) \mathrm{d}\lambda_m(t) = \sum_{i=0}^{m} \left[A_i g^{(i)}(0) + B_i g^{(i)}(1) \right] + \\ + \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu - 1} \lambda_{\nu\mu}^{(N)} g^{(\mu)}(\hat{t}_\nu^{(N)}) + R_{n,m}^{(N)}(g; \mathrm{d}\lambda_m),$$

where $d\lambda_m$ is the above specified measure, and $\hat{t}_\nu^{(N)}$, $\nu = \overline{1, n}$, are the zeros of the monic polynomial $\hat{\omega}_N$ (if it exists and it has the real zeros in the interval (0,1)), the remainder term verifies the relation

$$R_{n,m}^{(N)}(g\,;\,d\lambda_m) = 0, \text{ when } g \in \mathscr{P}_{N+n+2m+1}.$$

This quadrature formula is related to the following generalized Gauss-Christoffel quadrature formula

$$\int_0^1 g(t)d\,\sigma_m(t) = \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1} \sigma_{\nu\mu}^{(N)} g^{(\mu)}(\hat{t}_\nu^{(N)}) + R_n^{(N)}(g\,;\,d\,\sigma_m), \tag{3.4}$$

where $d\,\sigma_m(t) = t^{m+1}(1-t)^{m+1}d\lambda_m(t)$, the nodes $\hat{t}_\nu^{(N)}$, $\nu = \overline{1, n}$, are the same as in Gauss-Lobatto formula, and the weights from the two quadrature formulae are related by

$$\sigma_{\nu\mu}^{(N)} = \sum_{k=0}^{r_\nu-\mu-1} \lambda_{\nu,\mu+k}^{(N)} \binom{\mu+k}{k} \Big[t^{m+1}(1-t)^{m+1}\Big]_{t=\hat{t}_\nu^{(N)}}^{(k)},$$

$\nu = \overline{1, n}$, $\mu = \overline{0, r_\nu - 1}$. The remainder term satisfies the relation

$$R_n^{(N)}(g\,;\,d\,\sigma_m) = 0, \text{ when } g \in \mathscr{P}_{N+n-1}.$$

THEOREM 3.2. *If the conditions of Theorem 3.1. are satisfied, then the spline function (1.2) which solves the problem is given by*

$$t_\nu = \hat{t}_\nu^{(N)},\ \alpha_{\nu\mu} = \lambda_{\nu\mu}^{(N)},\ \nu = \overline{1, n},\quad \mu = \overline{0, r_\nu - 1},$$

and

$$\beta_i = B_{m-i} + \gamma_i,\quad i = \overline{0, m},$$

where $t_\nu^{(N)}$, $\nu = \overline{1, n}$, are the nodes of the generalized Gauss-Lobatto quadrature (or the nodes of corresponding generalized Gauss-Christoffel quadrature), B_i, $i = \overline{0, m}$, and $\lambda_{\nu\mu}^{(N)}$, $\nu = \overline{1, n}$, $\mu = \overline{0, r_\nu - 1}$, are the weights in generalized Gauss-Lobatto quadrature.

Proof. One takes in Gauss-Lobatto quadrature formula

$g(t) = t^{m+1}p(t)$, $p \in \mathscr{P}_{N+n+m}$, then $R_{n,m}^{(N)}(g\,;\,d\lambda_m) = 0$, and so that

$$\sum_{i=0}^{m} B_i\Big[\frac{d^i}{dt^i}(t^{m+1}p(t))\Big]_{t=1} + \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1}\lambda_{\nu\mu}^{(N)}\Big[\frac{d^\mu}{dt^\mu}(t^{m+1}p(t))\Big]_{t=\hat{t}_\nu^{(N)}} =$$

$$= \int_0^1 t^{m+1}p(t)d\lambda_m(t),$$

which is equivalent to

$$\sum_{i=0}^{m} B_i\Big[\frac{d^i}{dt^i}(t^{m+1+j})\Big]_{t=1} + \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1}\lambda_{\nu\mu}^{(N)}\Big[\frac{d^\mu}{dt^\mu}(t^{m+1+j})\Big]_{t=\hat{t}_\nu^{(N)}} =$$

$$= \int_0^1 t^{m+1+j}d\lambda_m(t),\quad j = \overline{0, N+n+m}.$$

If one compares these relations to relations (3.1) it results the assertion of the theorem.

Remark. If f is completely monotonic on [0,1], and the multiplities r_ν, $\nu = \overline{1, n}$, are odd positive integers then the measure $d\lambda_m$ and also $d\,\sigma_m$ are positive measure. In this case the existence and the uniqueness of the generalized Gauss-Christoffel quadrature are ensured, and so the corresponding Gauss-Lobatto quadrature [4,5].

4. Error of spline approximation. Further on the error term in the spline approximation formula

$$f(x) = s_{r,m}(x) + e_{r,m}(x)$$

is studied. The error term $e_{r,m}(x)$ is expressed in respect to the remainder terms of the generalized Gauss-Lobatto quadrature and Gauss-Christoffel quadrature.

THEOREM 4.1. *If the conditions of Theorem 3.1 are satified, then or any $x \in (0, 1)$ we have that*

$$e_{r,m}(x) = R_{n,}^{(N}(\rho_x\,;\,d\lambda_m) = R_n^{(N)}(\sigma_x\,;\,d\,\sigma_m),$$

where $R_{n,m}^{(N)}$ and $R_n^{(N)}$ are respectively the remainder terms in generalized Gauss-Lobatto quadrature (3.3) and generalized Gauss-Christoffel quadrature (3.4), and

$$\rho_x(t) = (t-x)_+^m,\quad \sigma_x(t) = \frac{\rho_x(t) - H_{2m+1}(\rho_x\,;\,t)}{t^{m+1}(1-t)^{m+1}},$$

with H_{2m+1} the Hermite interpolating polynomial relative to the function ρ_x and the nodes 0 and 1 with the same $m+1$ multiplicity.

Proof. By Taylor's formula we have

$$f(x) = \sum_{k=0}^{m}\frac{1}{k!}f^{(k)}(1)(x-1)^k + \frac{(-1)^{m+1}}{m!}\int_0^1 (t-x)_+^m f^{(m+1)}(t)dt =$$

$$= \sum_{k=0}^{m}\frac{1}{k!}f^{(k)}(1)(x-1)^k + \int_0^1 \rho_x(t)d\lambda_m(t).$$

On the other hand it is known that

$$s_{r,m}(x) = \sum_{k=0}^{m} \frac{1}{k!} p_m^{(k)}(1)(x-1)^k + \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu - 1} \lambda_{\nu\mu}^{(N)} \left[\frac{d^\mu}{dt^\mu} (t-x)_+^m \right]_{t=\hat{t}_\nu^{(N)}}$$

and so that

$$e_{r,m}(x) = \int_0^1 \rho_x(t) d\lambda_m(t) +$$

$$+ \sum_{k=0}^{m} \frac{m!}{k!} (\gamma_k - \beta_k)(1-x)^k - \sum_{\nu=1}^{n} \sum_{\mu=1}^{r_\nu - 1} \lambda_{\nu\mu}^{(N)} \rho_x^{(\mu)}(\hat{t}_\nu^{(N)}).$$

But

$$\beta_k - \gamma_k = B_{m-k}, \quad \rho_x^{(k)}(0) = 0, \quad \text{and} \quad \rho_x^{(k)}(1) = \frac{m!}{(m-k)!} (1-x)^{m-k},$$

and therefore

$$e_{r,m}(x) = \int_0^1 \rho_x(t) d\lambda_m(t) +$$

$$- \sum_{k=0}^{m} [A_k \rho_x^{(k)}(0) + B_k \rho_x^{(k)}(1)] - \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu - 1} \lambda_{\nu\mu}^{(N)} \rho_x^{(\mu)}(\hat{t}_\nu^{(N)}) =$$

$$= R_{n,m}^{(N)}(\rho_x ; d\lambda_m).$$

If one denotes $h(t) = \rho_x(t) - H_{2m+1}(\rho_x ; t)$, then $h^{(k)}(0) = h^{(k)}(1) = 0$, $k = \overline{0, m}$, we can write

$$R_{n,m}^{(N)}(h ; d\lambda_m) = R_n^{(N)} \left(\frac{h(t)}{t^{m+1}(1-t)^{m+1}} ; d\sigma_m \right). \tag{4.1}$$

But we have that

$$R_{n,m}^{(N)}(\rho_x ; d\lambda_m) = R_{n,m}^{(N)}(\rho_x - H_{2m+1}(\rho_x ; t) ; d\lambda_m) = \tag{4.2}$$

$$= R_{n,m}^{(N)}(h ; d\lambda_m).$$

From (4.1) and (4.2) it results that

$$e_{r,m}(x) = R_n^{(N)}(\sigma_x ; d\sigma_m).$$

5. Numerical examples. We have considered the same examples from [3], namely the exponential function e^{-ct}, $c > 0$, and trigonometric function $\sin \frac{\pi t}{2}$, $t \in [0, 1]$. In all cases quadratic spline approximation is presented.

Table 5.1 contains the elements of the quadratic spline function with a single knot t_1 of the multiplicity 3, which is written in the form

$$s_{(3),2}(t) = 2\beta_0 + 2\beta_1(1-t) + \beta_2(1-t)^2 +$$

$$+ \alpha_{10}(t_1 - t)_+^2 + 2\alpha_{11}(t_1 - t)_+ + 2\alpha_{12}(t_1 - t)_+^0.$$

Table 5.1

$f(t)$	β	α	t_1	M
$e^{-\frac{t}{10}}$	0.45242(+0) 0.45235(−1) 0.46139(−2)	0.28997(−3) −0.43930(−6) 0.30202(−5)	0.497727	0.30427(−5)
$e^{-\frac{t}{5}}$	0.40937(+0) 0.81824(−1) 0.17034(−1)	0.22083(−2) −0.66888(−5) 0.22992(−4)	0.495455	0.23335(−4)
$e^{-\frac{t}{2}}$	0.30328(+0) 0.15103(+0) 0.83810(−1)	0.29856(−1) −0.22554(−3) 0.31009(−3)	0.488641	0.32164(−3)
e^{-t}	0.18404(+0) 0.18064(+0) 0.22598(+0)	0.18954(+0) −0.28394(−2) 0.19515(−2)	0.477308	0.20965(−2)
e^{-2t}	0.68048(−1) 0.12293(+0) 0.41763(+0)	0.99067(+0) 0.28710(−1) 0.98568(−2)	0.454822	0.11307(−1)
$\sin \frac{\pi t}{2}$	0.50026(+0) −0.78003(−2) −0.11575(+1)	0.81049(+0) −0.17189(−1) 0.75443(−2)	0.460812	0.84100(−2)

Tables 5.2 and 5.3 contain the elements of the quadratic spline approximation function with two knots $t_1, t_2 \in (0,1)$, $t_1 < t_2$, first when t_1 has the multiplicity 3 and the knot t_2 is simply, and then the knot t_1 is simply and t_2 has the multiplicity 3. In the first case the spline function is written in the form

$$s_{(3,1),2}(t) = 2\beta_0 + 2\beta_1(1-t) + \beta_2(1-t)^2 +$$

$$+ \alpha_{10}(t_1 - t)_+^2 + \alpha_{11}(t_1 - t)_+ + \alpha_{12}(t_1 - t)_+^0 + \alpha_{20}(t_2 - t)_+^2$$

and in the second

$$s_{(1,3),2}(t) = 2\beta_0 + 2\beta_1(1-t) + \beta_2(1-t)^2 +$$

$$+ \alpha_{10}(t_1 - t)_+^2 + \alpha_{20}(t_2 - t)_+^2 + \alpha_{21}(t_2 - t)_+ + \alpha_{22}(t_2 - t)_+^0.$$

In the three tables we denoted $\beta = (\beta_0, \beta_1, \beta_2)'$, $M = \sup\{|f(t) - s_{.,2}(t)| ; 0 \leqslant t \leqslant 1\}$, and also in Table 5.1 we denoted $\alpha = (\alpha_{10}, \alpha_{11}, \alpha_{12})$, while in Tables 5.2 and 5.3 $\alpha = (\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{20})_+^0$ respectively $\alpha = (\alpha_{10}, \alpha_{20}, \alpha_{21}, \alpha_{22})'$.

All numerical experiments had been effectuated in double precision on the Romanian computer CORAL 4030.

Table 5.2

$f(t)$	β	α	$(t_1, t_2)^t$	M
$e^{-\frac{t}{10}}$	0.45252(+0) 0.42239(−1) 0.45832(−2)	0.12207(−3) 0.22762(−3) 0.15402(−5) 0.13979(−5)	0.360102 0.744821	0.14920(−5)
$e^{-\frac{t}{5}}$	0.40937(+0) 0.81852(−1) 0.16807(−1)	0.91016(−3) 0.17524(−2) 0.10192(−4) 0.10721(−4)	0.358334 0.743352	0.11377(−4)
$e^{-\frac{t}{2}}$	0.30327(+0) 0.15137(+0) 0.81008(−1)	0.11531(−1) 0.24460(−1) 0.73472(−4) 0.14781(−3)	0.353065 0.738896	0.15417(−3)
e^{-t}	0.18397(+0) 0.18255(+0) 0.21075(+0)	0.65400(−1) 0.16337(+0) −0.24953(−3) 0.96428(−3)	0.344405 0.731303	0.97705(−3)
e^{-2t}	0.67768(−1) 0.13037(+0) 0.36066(+0)	0.26871(+0) 0.93618(+0) −0.92435(−2) 0.52102(−2)	0.327569 0.715511	0.54515(−2)
$\sin\frac{\pi t}{2}$	0.50006(+0) −0.25681(−2) −0.11972(+1)	0.22910(+0) 0.72387(+0) −0.10882(−2) 0.39222(−2)	0.335672 0.702875	0.39658(−2)

Table 5.3

$f(t)$	β	α	$(t_1, t_2)^t$	M
$e^{-\frac{t}{10}}$	0.45242(+0) 0.45239(−1) 0.45864(−2)	0.12727(−3) 0.22266(−3) −0.19339(−5) 0.13773(−5)	0.252266 0.636346	0.14868(−5)
$e^{-\frac{t}{5}}$	0.40937(+0) 0.81850(−1) 0.16830(−1)	0.98928(−3) 0.16768(−2) −0.16186(−4) 0.10407(−4)	0.250822 0.634561	0.11298(−4)
$e^{-\frac{t}{2}}$	0.30327(+0) 0.15135(+0) 0.81285(−1)	0.14202(−1) 0.21907(−1) −0.27566(−3) 0.13722(−3)	0.246540 0.629175	0.15150(−3)
e^{-t}	0.18397(+0) 0.18242(+0) 0.21220(+0)	0.99150(−1) 0.13111(+0) −0.22986(−2) 0.83120(−3)	0.239570 0.620096	0.94352(−3)
e^{-2t}	0.67780(−1) 0.12990(+0) 0.36572(+0)	0.61512(+0) 0.60529(+0) −0.16621(−1) 0.38835(−2)	0.226255 0.601598	0.46505(−2)
$\sin\frac{\pi t}{2}$	0.50006(+0) −0.28310(−2) −0.11944(+1)	0.43832(+0) 0.52386(+0) −0.14494(−1) 0.30292(−2)	0.235500 0.596658	0.36730(−2)

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Received 10.III.1990

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