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SOME FUNCTIONAL EQUATIONS CONNECTED WITH QUADRATIC FORMS

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1. Let R be a commutative ring with identity and let M be a unitary R-module. A function $f: M \to R$ is a quadratic form if

(i) $f(rx) = r^2 f(x)$ for all $r \in R$, $x \in M$,

(ii) $F: M \times M \to R$ is bilinear, where

$$F(x, y) := f(x + y) - f(x) - f(y), \text{ for all } x, y \in M \tag{1.1}$$

(see e.g. Jacobson [6]).

It is easy to check that any function defined by (1.1) satisfies the following function equation (see [4]).

$$F(x, y) + F(x + y, z) = F(x, y + z) + F(y, z). \tag{1.2}$$

If the function $f: M \to R$ satisfies the parallelogram law

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.3}$$

and 2 is not a zero divisor in R, then F is bi-auditive (cf Jordan — von Neumann [7]).

From (1.1) and (1.3) it follows that F satisfies the partial homogeneity condition

$$F(x, -y) = -F(x, y) \tag{1.4}$$

K. Davison has proved the following.

Proposition 1. *Let R be a ring in which 2 is not a zero divisor. Let M be R-module and suppose $F: M \times M \to R$ satisfies (1.2) and (1.4) Then F is bi-additive.*

Let X be a complex vector space and $L: X \times X \to C$ a sesquilinear form, then a function

$$f(x) = L(x, x), \text{ for all } x \in X$$

satisfies the parallelogram law (1.3) and

$$f(\lambda x) = |\lambda|^2 f(x), \ \lambda \in \mathbb{C}, \ x \in X. \tag{1.7}$$

S. Kurepa [8] has shown that if a function $f: X \to \mathbb{C}$ satisfies the parallelogram law and (1.7) then there exists a function L such that

(1.8) $$f(x) = L(x, x), \text{ for all } x \in X.$$

2. Our aim is to study the functional equations (1.2) and (1.3) on the non-commutative structures.

THEOREM 1. *Let $(G,.)$ be an arbitrary group and $(H, +)$ an abelian group such that every equation $2a=2b$ in H can be concellated by 2.*

If $F: G \to H$ satisfies the following relations.

(2.1) $$F(x, y) + F(xy, z) = F(x, yz) + F(y, z), \ \forall\ x, y, z \in G.$$

and

(2.2) $$F(x, y^{-1}) = -F(x, y)$$

then the function F verifies the relation

(2.3) $$F(x, yz) = F(x, y) + F(x, z) \text{ and } F(xy, z) = F(x, z) + F(y, z)$$

for all $x, y, z \in G$ i.e. is "bi-additivity".

Proof. In (2.1) replace z by z^{-1} and using (2.2) we obtain $F(x, y) - F(xy, z) = F(x, yz^{-1}) - F(y, z)$.

Adding this equality with (2.1) we have

(2.3) $$2F(x, y) = F(x, yz) + F(x, yz^{-1}).$$

Define the function $g: G \to H$ by

(2.4) $$g(y) := F(x, y)$$

and (2.5) becomes

(2.5) $$2g(y) = g(yz) + g(yz^{-1})$$

Interchanging y by z we get

(2.6) $$2g(z) = g(zy) + g(zy^{-1})$$

From relations (2.5) and (2.6) and using (2.2) and (2.4) we obtain the functional equation $f: G \to H$,

(2.7) $$g(yz) + g(zy) = 2[g(y) + g(z)], \ \forall\ y, z \in G$$

We need the following theorem (see [2]).

THEOREM 2. *Consider the functional equation (2.7), where $(G,.)$ is a arbitrary group and $(H, +)$ is a abelian group in which $2x = 0$ implies that $x = 0$.*

If $f: G \to H$ is a solution of equation (2.7), then f is a homomorphism of G into H.

Therefore we have

$$g(yz) = g(y) + g(z), \text{ for all } y, z \in G.$$

From this equality and (2.4) we obtain

(2.8) $$F(x, yz) = F(x, y) + F(x, z)$$

which together with (2.1) leads to

(2.9) $$F(xy, z) = F(x, z) + F(y, z)$$

This complet the proof of the Theorem

3. Consider the functional equation

(3.1) $$f: G \to H, \ f(xy) + f(xy^{-1}) = 2\ [f(x) + f(y)]$$

where $(G,.)$ is an arbitrary group and $(H, +)$ is an abelian group such that every equation $2a = 2b$ in H can be cancellated by 2.

The functional equation (31) was studied by J. Aczel in [1] where G and H are abelian groups and by M. Hosszú and M. Csikós in [5] where G is a finitely generated group and H an abelian group.

Equation (3.1) was studied in [3] where G is an arbitrary group and H is an abelian group. We prove it here for the convenience of a reader.

It is easy to see that

(3.2) $$f(e) = 0$$

(3.3) $$\forall\ y \in G, \ f(y) = f(y^{-1}).$$

(3.4) $$\forall\ x, y\ G, \ f(xy) = f(yx),$$

We prove by induction that

(3.5) $$f(x^n) = n^2 f(x).$$

Indeed, taking $y = x$ in (3.1) and in view of (3.2) we have

$$f(x^2) = 4f(x)$$

therefore (3.5) is true for $n = 2$. Suppose that (3.5) is valid for every natural number less than n.

Putting $y = x^{n-1}$ in (1), we have

$$f(x^n) + f(x^{2-n}) = 2f(x) + 2f(x^{n-1})$$

from the induction hypothesis and this relation, we have

$$f(x^n) = 2f(x) + 2(n-1)^2 f(x) - (n-2)^2 f(x) = n^2 f(x).$$

Define the function $F: G \times G \to H$ by

(3.7) $$2F(x, y) = f(xy) - f(x) - f(y).$$

From (3.4) and (3.7), we obtain

(3.8) $$F(x, y) = F(y, x)$$

THEOREM 3. *If $f: G \to H$ is a solution of equation (3.1) then $f(x) = F(x, x)$ where F is defined by (3.7) and it verifies the relation (3.8) and the following relation*

$$F(x, yz) = F(x, y) + F(x, z), \text{ for all } x, y, z \in G \tag{3.9}$$

Proof. We first prove the following

$$F(x, yz) + F(x, zy) = 2[F(x, y) + F(x, z)]. \tag{3.10}$$

From (3.7), (3.1) and (3.4) we obtain

$$\begin{aligned} 4[F(x, y) + F(x, z)] &= 2[f(xy) + f(xz) - 2f(x) - f(y) - f(z)] = \\ &= 2f(xz) + 2f(y) - f(zxy^{-1}) + f(zxy^{-1}) - 4f(y) + 2f(xy) - \\ &- 4f(x) - 2f(z) = f(xyz) + f(zxy^{-1}) - 4f(x) - 4f(y) + 2f(xy) - \\ &- 2f(z) = f(xyz) + f(zxy^{-1}) - 2f(xy^{-1}) - 2f(z) = \\ &= f(xyz) - f(zyx^{-1}) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} 4F(x, yz) &= 2f(xyz) - 2f(x) - 2f(yz) = \\ &= 2f(xyz) - f(xyz) - f(x^{-1}yz) = f(xyz) - f(x^{-1}yz). \end{aligned} \tag{3.12}$$

From (3.12), (3.1), (3.4) and (3.11) we have

$$\begin{aligned} 4[F(x, yz) + F(x, zy)] &= f(xyz) - f(x^{-1}yz) + f(xzy) - f(x^{-1}zy) = \\ &= f(xyz) - f(zyx^{-1}) - [2f(x) + 2f(yz) - f(xyz)] + \\ &+ [2f(x) + 2f(zy) - f(x^{-1}zy)] = 2[f(xyz) - f(zyx^{-1})] = \\ &= 8[F(x, y) + F(x, z)]. \end{aligned}$$

and the relation (3.10) holds.

For x fixed, $x \in G$, define the function $g: G \to H$

$$g(y) = F(x, y). \tag{3.13}$$

Then, from (3.10) and (3.13) it follows

$$g(yz) + g(zy) = 2[g(y) + g(z)]. \tag{3.14}$$

Now, using Theorem 2 it follows that g is a homomorphism of $(G,.)$ into $(H, +)$ and (3.9) holds.

4. Now we will show some properties of the function F defined by (3.7).

By X and X' we denote complex vector spaces and by A complex algebra with unit 1 we assume that the algebra A has the following regularity.

For any $t \in A$ there exists a natural number n such that $t + n$ and $t + n + 1$ are invertible elements in A.

Furthermore we assume that X is a left module over A and X' is left and right modul orer A.

THEOREM 4. *Let A, X and X' be as above. If $f: X \to X'$ is a solution of the parallelogram law*

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \text{ for all } x, y \in X \tag{4.1}$$

and if f satisfies the homogeneity condition

$$f(tx) = tf(x)t, \ t \in A, \ x \in X \tag{4.2}$$

then the function $F: X \times X \to X'$ defined by

$$2F(x, y) = f(x + y) - f(x) - f(y) \tag{4.3}$$

is additive, symmetric, $f(x) = F(x, x)$ and

$$F(tx, y) + F(x, ty) = tF(x, y) + F(x, y)t, \ t \in A; \ x, y \in X \tag{4.4}$$

Furthermore the function

$$h(t; x, y) = F(tx, y) - F(x, ty), \ t \in A, x \in X$$

is a Jordan derivation on A, i.e.

$$h(t \circ s; \ x, y) = t \circ h(s; \ x, y) + h(t; \ x, y) \circ s \tag{4.5}$$

holds, where

$$t \circ s = ts + st, \ t, \ a \in A$$

Proof. From Theorem 3 follows that the function F defined by (4.3) is bi-additive, symmetric and $f(x) = F(x, x)$ for all $x \in X$.

In the hypotheses of Theorem 4 it is easy to check that the function F can be reduced to the function M (see [8]).

$$M(x, y) = (f(x + y) - f(x - y))/8 - i(f(x + iy) - f(x - iy)/8, \quad \forall x, \ y \in X,$$

therefore (4.4) and (4.5) are true.

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