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THEOREM 3, a Let E day a margned linear spaces Then E is strict A GENERALIZATION OF JAMES' AND KREIN'S THEOREMS weren that to I ai thomash

We recall now a characterization of strict convex spaces in terms of maximal elements dues to M.G. Krein (see for example [83, pr. 1023];

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(Băile Herculane)

(i) B is strict convex (vottering (reflexive and strict convex)); 1. Introduction. Let E be a real normed linear space and consider the following semi-inner products (,),, (,), defined on E and given by $(x,y)_t := \lim_{t \to \infty} (\|y + tx\|^2 - \|y\|^2)/2t, \ x, \ y \in E;$

$$(x,y)_i := \lim_{t \to 0-} (\|y + tx\|^2 - \|y\|^2)/2t, \ x, y \in E;$$

$$(x, y)_s := \lim_{t \to 0+} (\|y + tx\|^2 - \|y\|^2)/2t, \ x, y \in E;$$

$$(x, y)_s := \lim_{t \to 0+} (\|y + tx\|^2 - \|y\|^2)/2t, \ x, y \in E.$$

For the sake of completeness we list some usual properties of these semi-inner products that will be used in the sequel (see also [2]):

- (i) $(x, x)_p = ||x||^2$ for all x in E; (ii) $(-x, y)_s = (x, -y)_s = -(x, y)_s$ if $(x, y \in E;$
- (iii) $(\alpha x, \beta y)_p = \alpha \beta(x, y)_p$ for all $x, y \in E$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \beta \geq 0$;
- (iv) $(\alpha x + y, |x)_p = \alpha ||x||^2 + (y, |x)_p$ for all $x, y \in E$ and $\alpha \in \mathbb{R}$;
 - (v) $(x + y, z)_{p} \le ||x|| ||z|| + (y, z)_{p}$ for all $x, y, z \in E$;
- (vi) the element $x \in E$ is Birkhoff orthogonal over $y \in E$, i.e., ||x|| + ||x|| $+ ty \parallel \geqslant \|x\|$ for all $t \in \mathbb{R}$ and we denote $x \perp y$ iff $(y, x)_t \leqslant 0 \leqslant (y, x)_s$;
- (vii) the space E is smooth iff $(x, y)_i = (x, y)_s$ for all x, y in E or iff () is linear in the first variable: iff $(,)_p$ is linear in the first variable; where p = s or p = i.

For some properties of $(,)_n$ in connection with best approximation and continuous linear functionals, see [2] where further details are given.

To recall some well-known theorems of reflexivity and strict convexity due to R.C. James and M.G. Krein, respectively, we need the following concept: the nonzero element $u \in E$ is a maximal element for the functional $\bar{f} \in E^* \setminus \{0\}$ if tinuous linear tunctional on W. Then, by Theore

 $\inf f(u) = \lim \|f\|_{\mathcal{D}} \|u\| \perp_u w \text{ and } \|a = \{0\}/A =$

(see also [8], p. 33); In not () and R w(an) - out)

THEOREM 1. ([5)] Let E be a Banach space. Then E is reflexive iff every nonzero continuous linear functional on E has at least one maximal element in E.

Another famous result of R.C. James is the following.

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THEOREM 2: ([6]) Let E be a Banach space. E is reflexive iff for every closed and homogeneous hyperplane H in E there exists a point u e $\in E \setminus \{0\}$ such that $u \perp H$.

We recall now a characterization of strict convex spaces in terms of maximal elements dues to M.G. Krein (see for example [8], p. 102).

THEOREM 3. Let E be a normed linear space. Then E is strict convex iff every nonzero continuous linear functional on E has at most one element in E of the norm one. MARCHIE

Recently, we proved the following result (see [3], p. 384):

THEOREM 4. Let E be a normed [(Banach)] space. Then the following sentences are equivalent

(i) E is strict convex [reflexive (reflexive and strict convex)];

(ii) for every G a [(closed)] linear subspace in E and for each x in E there exists at most one [at least one a (a unique)] x' in G and at most one [at least one (a unique)] element x'' in G such that x = x' + x'', where G1 denotes the orthogonal complement of G in the sense of Birkhoff.

It is clear that this result contains Theorem 2 of R.C. James and

gives a similar characterization for strict convex spaces,

2. Main results. The following result improves James' and Krein's theorems for the case of real normed linear spaces.

THEOREM 5. Let E be a real normed [(Banach)] space. Then the following statements are equivalent:

(i) E is strict convex [reflexive (reflexive and strict convex)];

(ii) for every nonzero continuous linear functional f on E there exists at most one [at least one (a unique)] element u in E, ||u|| = 1 such that the following interpolation holds: (2 M) + (2 M > (2 M + 1) (7)

(1) ||f|| = f(x) and ||f|| = f(x) ||f|| = f(x)for all x in E. The was a sound on has H at Ha not has s we +

Proof. "(i) \Rightarrow (ii)". a. Assume that E is strict convex, f is a nonzero continuous linear functional on E and suppose, by absurd, that there exists two distinct elements $u, v \in E \setminus \{0\}, ||u|| = ||v|| = 1$ such that (1) holds. Then f(u) = f(v) = ||f||, i.e., u, v are maximal elements of the norme one, which contradicts Krein's theorem.

"(ii)⇒(i)". a. The converse implication is also obvious from Krein's

theorem. We shall omit the details.

"(i) \Rightarrow (ii)". b. Suppose that E is reflexive and let f be a nonzero continuous linear functional on E. Then, by Theorem 2, there exists $w_0 \in$ $\in E \setminus \{0\}$ such that $w_0 \perp \operatorname{Ker}(f)$, and since

$$f(x)w_0 - f(w_0)x \in \text{Ker}(f)$$
 for all x in E [8] only have

we derive, by (vi), that a down of mod M fall (G) I MERONIA $(f(x)w_0 - f(w_0)x, \ w_0)_i \leqslant 0 \leqslant (f(x)w_0 - f(w_0)x, \ w_0)_s$

Another famous result of R.C. James is the following number of

By the use of semi-inner products properties (see (i) - (iv)) we have :

$$\frac{f(w_0)}{\|w_0\|} \left(x, \frac{w_0}{\|w_0\|} \right)_i \le f(x) \le \frac{f(w_0)}{\|w_0\|} \left(x, \frac{w_0}{\|w_0\|} \right)_s \text{ for all } x \in E$$
if $f(w_0) > 0$ and

6. Lames E. C. Reflectiony and the suprement of these functionals base 0 €n(ov) this

$$\frac{1}{\|w_0\|} \frac{f(w_0)}{\|w_0\|} \left(x, -\frac{w_0}{\|w_0\|}\right)_i \leqslant f(x) \leqslant \frac{f(w_0)}{\|w_0\|} \left(x, -\frac{w_0}{\|w_0\|}\right)_s \text{ for all } x \in E$$
 if $f(w_0) < 0$.

On the other hand, a simple calculus shows that we have

$$\|f\|=\|f(w_0)/\|w_0\|.$$

Consequently, putting $u:=w_0/\|w_0\|$ if $f(w_0)>0$ or $u:=-w_0/\|w_0\|$ if $f(w_0)>0$ or $u:=-w_0/\|w_0\|$ $||w_0||$ if $f(w_0) < 0$, we conclude that the interpolation (1) holds.

"(ii) \Rightarrow (i)". b. The converse implication is obvious from Theorem 1 of R.C. James and we shall omit the details.

"(i) \Rightarrow (ii)". c. The statement : E is reflexive and strict convex iff E is a Banach space with the property that for all nonzero continuous linear functional f on E there exists a unique element u in E, ||u|| = 1 such that (1) holds, is proven by the above arguments.

Corollary 1. Let E be a smooth real normed (Banach) space. Then E is strict convex [reflexive (reflexive and strict convex)] if and only if for every nonzero continuous linear functional f on E there exists at most one [at least one (a unique)] element u in E, ||u||=1 such that the following representation holds:

(2)
$$f(x) = ||f|| (x, u)_s \text{ for all } x \text{ in } E.$$

COROLLARY 2. Let E be a smooth complex normed [(Banach)] space. Then E is strict convex [reflexive (reflexive and strict convex)] iff for every nonzero continuous linear functional f on E there exists at most one [at least one (a unique)] element u in E, ||u|| = 1 such that the following representation holds:

(3)
$$f(x) = ||f|| [(x, u)_s - i(ix, u)_s] for all x in E.$$

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The proof is obvious from Theorem 5 and we shall omit the details.

Remark. The above corollaries improve: Theorem 6 of J.R. Giles [4], Proposition 2 bis of P.L. Papini [7] and Theorem 1.8 from [1]. Remark. It is $L_{\theta}(J_1,0) = L_{\theta+1}(J_2,0) = f(0)$ for all ϕ .

A. Pupus had also proved in [3] the relation REFERENCES

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(ii) (c. The sintennent: L is reflexive and short convex the is a Banach space with the property that for all nonzero continuous linear functional t on E there exists a conque element a in E, (v) = 1, such that (1) holds; is printed by the above expansion).

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