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A GENERALIZATION OF JAMES' AND KREIN'S THEOREMS

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1. Introduction. Let E be a real normed linear space and consider the following semi-inner products $(,)_i$, $(,)_s$ defined on E and given by

$$(x,y)_i := \lim_{t\to 0-} (\|y+tx\|^2 - \|y\|^2)/2t, \quad x,\ y \in E;$$

$$(x,y)_s := \lim_{t\to 0+} (\|y+tx\|^2 - \|y\|^2)/2t, \quad x,\ y \in E.$$

For the sake of completeness we list some usual properties of these semi-inner products that will be used in the sequel (see also [2]):

(i) $(x,x)_p = \|x\|^2$ for all x in E;

(ii) $(-x,y)_s = (x,-y)_s = -(x,y)_i$ if $x,\ y \in E$;

(iii) $(\alpha x, \beta y)_p = \alpha\beta(x,y)_p$ for all $x,\ y \in E$ and $\alpha,\ \beta \in \mathbb{R}$, $\alpha\beta \geq 0$;

(iv) $(\alpha x + y, x)_p = \alpha\|x\|^2 + (y,x)_p$ for all $x,\ y \in E$ and $\alpha \in \mathbb{R}$;

(v) $(x+y, z)_p \leq \|x\|\ \|z\| + (y,z)_p$ for all $x,\ y,\ z \in E$;

(vi) the element $x \in E$ is Birkhoff orthogonal over $y \in E$, i.e., $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$ and we denote $x \perp y$ iff $(y,x)_i \leq 0 \leq (y,x)_s$;

(vii) the space E is smooth iff $(x,y)_i = (x,y)_s$ for all $x,\ y$ in E or iff $(,)_p$ is linear in the first variable;

where $p = s$ or $p = i$.

For some properties of $(,)_p$ in connection with best approximation and continuous linear functionals, see [2] where further details are given.

To recall some well-known theorems of reflexivity and strict convexity due to R.C. James and M.G. Krein, respectively, we need the following concept: the nonzero element $u \in E$ is a maximal element for the functional $f \in E^* \setminus \{0\}$ if

$$f(u) = \|f\|\ \|u\|$$

(see also [8], p. 33).

THEOREM 1. ([5]) *Let E be a Banach space. Then E is reflexive iff every nonzero continuous linear functional on E has at least one maximal element in E.*

Another famous result of R.C. James is the following.

THEOREM 2. ([6]) *Let E be a Banach space. E is reflexive iff for every closed and homogeneous hyperplane H in E there exists a point $u \in E \setminus \{0\}$ such that $u \perp H$.*

We recall now a characterization of strict convex spaces in terms of maximal elements dues to M.G. Krein (see for example [8], p. 102).

THEOREM 3. *Let E be a normed linear space. Then E is strict convex iff every nonzero continuous linear functional on E has at most one element in E of the norm one.*

Recently, we proved the following result (see [3], p. 384):

THEOREM 4. *Let E be a normed [(Banach)] space. Then the following sentences are equivalent*

(i) *E is strict convex [reflexive (reflexive and strict convex)];*

(ii) *for every G a [(closed)] linear subspace in E and for each x in E there exists at most one [at least one a (a unique)] x' in G and at most one [at least one (a unique)] element x'' in G such that $x = x' + x''$, where $G^{\perp}$ denotes the orthogonal complement of G in the sense of Birkhoff.*

It is clear that this result contains Theorem 2 of R.C. James and gives a similar characterization for strict convex spaces.

2. Main results. The following result improves James' and Krein's theorems for the case of real normed linear spaces.

THEOREM 5. *Let E be a real normed [(Banach)] space. Then the following statements are equivalent:*

(i) *E is strict convex [reflexive (reflexive and strict convex)];*

(ii) *for every nonzero continuous linear functional f on E there exists at most one [at least one (a unique)] element u in E, $\|u\| = 1$ such that the following interpolation holds:*

(1) $$\|f\|(x, u)_i \leq f(x) \leq \|f\|(x, u)_s$$

for all x in E.

Proof. "(i)⇒(ii)". a. Assume that E is strict convex, f is a nonzero continuous linear functional on E and suppose, by absurd, that there exists two distinct elements $u, v \in E\setminus\{0\}$, $\|u\| = \|v\| = 1$ such that (1) holds. Then $f(u) = f(v) = \|f\|$, i.e., u, v are maximal elements of the norme one, which contradicts Krein's theorem.

"(ii)⇒(i)". a. The converse implication is also obvious from Krein's theorem. We shall omit the details.

"(i)⇒(ii)". b. Suppose that E is reflexive and let f be a nonzero continuous linear functional on E. Then, by Theorem 2, there exists $w_0 \in E\setminus\{0\}$ such that $w_0 \perp \mathrm{Ker}(f)$, and since

$$f(x)w_0 - f(w_0)x \in \mathrm{Ker}(f) \text{ for all } x \text{ in } E$$

we derive, by (vi), that

$$(f(x)w_0 - f(w_0)x,\ w_0)_i \leq 0 \leq (f(x)w_0 - f(w_0)x,\ w_0)_s$$

for all x in E.

By the use of semi-inner products properties (see (i)–(iv)) we have:

$$\frac{f(w_0)}{\|w_0\|}\left(x, \frac{w_0}{\|w_0\|}\right)_i \leq f(x) \leq \frac{f(w_0)}{\|w_0\|}\left(x, \frac{w_0}{\|w_0\|}\right)_s \text{ for all } x \in E$$

if $f(w_0) > 0$ and

$$\frac{f(w_0)}{\|w_0\|}\left(x, -\frac{w_0}{\|w_0\|}\right)_i \leq f(x) \leq -\frac{f(w_0)}{\|w_0\|}\left(x, -\frac{w_0}{\|w_0\|}\right)_s \text{ for all } x \in E$$

if $f(w_0) < 0$.

On the other hand, a simple calculus shows that

$$\|f\| = |f(w_0)|/\|w_0\|.$$

Consequently, putting $u := w_0/\|w_0\|$ if $f(w_0) > 0$ or $u := -w_0/\|w_0\|$ if $f(w_0) < 0$, we conclude that the interpolation (1) holds.

"(ii)⇒(i)". b. The converse implication is obvious from Theorem 1 of R.C. James and we shall omit the details.

"(i)⇒(ii)". c. The statement: E is reflexive and strict convex iff E is a Banach space with the property that for all nonzero continuous linear functional f on E there exists a unique element u in E, $\|u\| = 1$ such that (1) holds, is proven by the above arguments.

COROLLARY 1. *Let E be a smooth real normed (Banach) space. Then E is strict convex [reflexive (reflexive and strict convex)] if and only if for every nonzero continuous linear functional f on E there exists at most one [at least one (a unique)] element u in E, $\|u\| = 1$ such that the following representation holds:*

(2) $$f(x) = \|f\|\,(x, u)_s \text{ for all } x \text{ in } E.$$

COROLLARY 2. *Let E be a smooth complex normed [(Banach)] space. Then E is strict convex [reflexive (reflexive and strict convex)] iff for every nonzero continuous linear functional f on E there exists at most one [at least one (a unique)] element u in E, $\|u\| = 1$ such that the following representation holds:*

(3) $$f(x) = \|f\|\,[(x, u)_s - i(ix, u)_s] \text{ for all } x \text{ in } E.$$

The *proof* is obvious from Theorem 5 and we shall omit the details.

Remark. The above corollaries improve: Theorem 6 of J.R. Giles [4], Proposition 2 bis of P.L. Papini [7] and Theorem 1.8 from [1].

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