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BASKAKOV OPERATORS AND CONVEX FUNCTIONS

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Abstract. The aim of this note is to establish a relation between the sequence formed by Baskakov operators and convex functions.

Let $Q[a, b]$ be the set of real functions defined and bounded in the interval $[0, \infty)$ and continuous in the interval $[a, b]$, continuous to the left in $x = a$ and continuous to the right in $x = b$.

The papers [1], [3], [4] and some others deal with the following class of linear positive operators (so called *Baskakov operators*)

(1) $$L_n(f, x) = (1 - x)^n \sum_{\nu=0}^{\infty} \binom{n + \nu - 1}{\nu} x^\nu f\left(\frac{\nu}{n}\right), \quad n = 1, 2, \ldots$$

It is proved (for instance in [3] that for $f \in Q[0, a^*]$, $a^* = a/(1 - a)$, $0 < a < 1$, the sequence $\{L_n(f, x)\}$ converges uniformly towards the function $f(x/(1 - x))$ in the interval $[0, a]$.

We remind the following relations

(2) $$\left.\begin{aligned} L_n(1, x) &= 1 \\ L_n(t, x) &= \tau(x) \\ L_n(t^2, x) &= \tau^2(x) + \frac{x\tau'(x)}{n}, \end{aligned}\right\}$$

where $\tau(x) = x/(1 - x)$.

In [1], [3] there is proved the theorem:

THEOREM. *If the function f is defined in the interval $[0, \infty)$ and if f is convex in this interval, then the sequence $\{L_n(f, x)\}$ is decreasing in the interval* $(0, a]$, i.e.

$$L_n(f, x) > L_{n-1}(f, x), \quad x \in (0, a], \quad n = 1, 2, \ldots$$

Remark. It is $L_n(f, 0) = L_{n+1}(f, 0) = f(0)$ for all n.

A. Lupas has also proved in [3] the relation

$$\Delta_1 L_n(f, x) = L_{n+1}(f, x) - L_n(f, x)$$

(3) $$= (1 - x)^n \sum_{\nu=1}^{\infty} \Lambda_{\nu n}[f]\, x^\nu,$$

where

$$\Lambda_{\nu n}[f] = -\frac{1}{n(n+1)}\binom{n+\nu}{\nu-1}\left[\frac{\nu-1}{n+1}, \frac{\nu}{n+1}, \frac{\nu}{n}; f\right]$$

and $\left[\frac{\nu-1}{n+1}, \frac{\nu}{n+1}, \frac{\nu}{n}; f\right]$ denotes the divided difference of the function f. Now, we shall deal with the functions $T_{n,2}$ which are defined as follows

$$T_{n,2}(x) = (1-x)^n \sum_{\nu=0}^{\infty}\binom{n+\nu-1}{2} x^\nu(\nu - n\tau(x))^2 \tag{4}$$

By using (2) it is easy to show that

$$T_{n,2}(x) = nx\tau'(x).$$

Using (4) we can obtain the useful estimation ($\delta > 0$):

$$(1-x)^n \sum_{\left|\frac{\nu}{n}-\tau(x)\right| > \delta}\binom{n+\nu-1}{\nu} x^\nu \leqq$$

$$\leqq \frac{(1-x)^n}{\delta^2}\sum_{\nu=0}^{\infty}\binom{n+\nu-1}{\nu} x^\nu\left(\frac{\nu}{n}-\tau(x)\right)^2 \leqq \frac{T_{n,2}}{n^2\delta^2} \leqq \frac{x\tau'(x)}{n\delta^2} \tag{5}$$

For sufficiently large n we can put $\delta = n^{-\alpha}$, $0 < \alpha < \frac{1}{2}$, and then

$$(1-x)^n \sum_{\left|\frac{\nu}{n}-\tau(x)\right|}\binom{n+\nu-1}{\nu} x^\nu \leqq x\tau'(x)\, n^{-\beta},\ \beta > 0. \tag{6}$$

This means that

$$(1-x)^n \sum_{\left|\frac{\nu}{n}-\tau(x)\right| < \delta}\binom{n+\nu-1}{\nu} x^\nu \to 1 \tag{7}$$

for $n \to \infty$.

Now we can prove the following theorem.

THEOREM. *Let $f \in Q[0, a^*]$ and let f'' be continuous and bounded in the interval $[0, \infty)$. Let the sequence $\{L_n(f, x)\}$ be decreasing in $(0, a]$. Then f is convex in the interval $[0, a^*]$.*

Proof. Let us suppose that f is not convex in the interval $[0, a^*]$. Then there is a point $\tau(x_0) \in (0, a^*)$ for which $f''(\tau(x_0)) < 0$.

We know it is valid

$$\left[\frac{\nu-1}{n+1}, \frac{\nu}{n+1}, \frac{\nu}{n}, f\right] = \frac{1}{2} f''(\xi_\nu),\ \xi_\nu \in \left(\frac{\nu-1}{n+1}, \frac{\nu}{n}\right).$$

And now we obtain for $\Lambda_1 L_n$

$$\Delta_1 L_n(f, x_0) = -\frac{(1-x_0)^n}{2n(n+1)}\sum_{\nu=1}^{\infty}\binom{n+\nu}{\nu-1} f''(\xi_\nu).$$

Because f'' is continuous in the interval $[0, a^*]$, it follows: there exists a positive number δ that for all $\tau(x) \in (\tau(x_0) - \delta, \tau(x_0) + \delta)$ it is $f''(\tau(x)) < -A$, $A > 0$.

Now

$$\Delta_1 L_n(f, x_0) = -\frac{(1-x_0)^n}{2n(n+1)}\sum_E\binom{n+\nu}{\nu-1} f''(\xi_\nu) - \tag{8}$$

$$-\frac{(1-x_0)^n}{2n(n+1)}\sum_{I-E}\binom{n+\nu}{\nu-1} f''(\xi_\nu),$$

where

$$I = \{\nu \mid \nu = 1, 2, \ldots\}$$

$$E = \left\{\nu \,\middle|\, \left|\frac{\nu-1}{n+1} - \tau(x)\right| < \delta, \left|\frac{\nu}{n} - \tau(x)\right| < \delta\right\}.$$

We denote the first sum by Λ_1 and the second one by Λ_2.

If is $f''(x) < -A$ on the set E and then we obtain for the sum Λ_1

$$\Lambda_1 \geqq \frac{A(1-x_0)^n}{2n(n+1)}\sum_E\binom{n+\nu}{\nu-1} x_0^\nu.$$

Putting $\mu = \nu - 1$ we have

$$\Lambda_1 \geqq \frac{A(1-x_0)^n}{2n(n+1)}\sum_{\substack{\left|\frac{\mu}{n+1}-\tau(x_0)\right| < \delta \\ \left|\frac{\mu-1}{n}-\tau(x_0)\right| < \delta}}\binom{n+\mu+1}{\mu} x_0^{\mu+1}$$

Taking into account (7) we can conclude

$$\Lambda_1 \geqq \frac{A(1-x_0)^n}{2(n+1)\,n} + 0\left(\frac{1}{n(n+1)}\right) \tag{9}$$

Let us estimate the second sum Λ_2 (let us put $|f''(x)| \leqq M$)

$$|\Lambda_2| \leqq \frac{(1-x_0)^n}{2(n+1)n}\sum_{I-E}\binom{n+\nu}{\nu-1} x_0^\nu M \leqq$$

$$\leqq \frac{M(1-x_0)^n}{2(n+1)n}\left(\sum_{\left|\frac{\nu-1}{n+1}-\tau(x_0)\right| \geqq \delta}\binom{n+\nu}{\nu-1} x_0^\nu\right) +$$

$$+\frac{M(1-x_0)^n}{2(n+1)\,n}\sum_{\left|\frac{\nu}{n}-\tau(x_0)\right| \geqq \delta}\binom{n+\nu}{\nu-1} x_0^\nu$$

Using the expressions for $T_{n2'} \times T_{n+1,2}$ and estimation (6) we obtain for sufficiently large n

$$|\Lambda_2| \leq \frac{Mx_0\tau'(x_0)}{2n(n+1)}\left(\frac{1}{(n+1)^{1-2\alpha}} + \frac{1}{n^{1-2\alpha}}\right) = 0\left(\frac{1}{n(n+1)}\right)$$

where $0 < \alpha < \frac{1}{2}$. From here and from (9) it follows (for sufficiently large n)

$$\Lambda_1 > |\Lambda_2|,$$

and then $\Delta_1 L_n(f, x_0) = L_{n+1}(f, x_0) - L_n(f, x_0) > 0$, i.e. the sequence $\{L_n(f, x_0)\}$ is not decreasing. This contradiction proves the theorem.

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