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METHOD FOR SOLVING THE PERIODIC PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In the paper a monotone-iterative method for approximate finding a couple a minimal and maximal quasisolutions of the periodic problem for a system of integro-differential equations of Volterra type is justified.

In the simulation of various processes in ecology, radio engineering, physics, etc. an important role is played by the periodic problems for systems of integro-differential equations. Such methods for a solving them, where not only the existence of the solutions is established but an iterative procedure for approximately finding them is proposed as well, are of particular interest. One of the most effective approaches in this respect is the monotone-iterative technique proposed by V. Lakshmikantham and elaborated for various classes of differential equations by his school [1]–[7].

In the present paper a modification of the monotone-iterative method for construction of quasisolutions of the periodic problem for systems of integro-differential equations is justified.

Consider the periodic problem for the system of integro-differential equations

$$\begin{aligned} \dot{x} &= f(t, x, Gx) && \text{for } t \in [0, T] \\ x(t) &= x(t+T) && \text{for } t \in [-T, 0] \end{aligned} \tag{1}$$

where $x \in R^n$, $x = (x_1, x_2, \ldots, x_n)$, $f: [0, T] \times R^n \times R^n \to R^n$, $Gx = (G_1x, G_2x, \ldots, G_nx)$, $G_ix = \int_{t-h}^{t} k_i(t, s)x_i(s)ds$, $(i = \overline{1, n})$, $k_i: [0, T] \times [-h, T]$, $T = \text{const} > 0$, $h = \text{const} > 0$, $h < T$.

With each positive integer $j = \overline{1, n}$ associate two positive integers p_j and q_j such that $p_j + q_j = n - 1$.

Introduce the notation

$$(x_j, [x]_{p_j}, [y]_{q_j}) = \begin{cases} (x_1, x_2, \ldots, x_{p_j}, y_{p_j+1}, y_{p_j+2}, \ldots, y_n) & \text{for } p_j \geqslant \\ (x_1, x_2, \ldots, x_{p_j}, y_{p_j+1}, \ldots, y_{j-1}, x_j, y_{j+1}, \ldots, y_n) & \text{for } p_j < j \end{cases}$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$.

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Remark 1. In the case when $p_j = j - 1$, then

$$(y_j,\ [x]_{p_j},\ [y]_{q_j}) = (x_1, x_2, \ldots, x_{j-1}, y_j, \ldots, y_n).$$

Definition 1. The functions $v, w : [-T, T] \to R^n$ are called a couple of lower and upper quasisolutions of period problem (1) if

$$\dot{v}_j \leqslant f_j(t, v_j, [v]_{p_j}, [w]_{q_j}, G_j v, [Gv]_{p_j}, [Gw]_{q_j}) \text{ for } t \in [0, T] \tag{2}$$

$$\dot{w}_j \geqslant f_j(t, w_j, [w]_{p_j}, [v]_{q_j}, G_j w, [Gw]_{p_j}, [Gv]_{q_j})$$

$$v_j(t) = v_j(t + T) \text{ for } t \in [-T, 0], \ (j = \overline{1, n}).$$

$$w_j(t) = w_j(t + T) \tag{3}$$

Definition 2. The functions $v, w : [-T, T] \to R^n$ are called a couple of quasisolutions of periodic problem (1) if relations (2) and (3) are fulfilled only as equalities.

Definition 3. The couple of quasisolutions $v, w : [-T, T] \to R^n$ of periodic problem (1) is called a couple of minimal and maximal quasisolutions if for any couple of quasisolutions $(u, \bar{u})$ of problem (1) the inequalities $v(t) \leqslant u(t) \leqslant w(t)$ and $v(t) \leqslant \bar{u}(t) \leqslant w(t)$ hold for $t \in [-T, T]$.

For any couple of functions $v, w : -[T, T] \to R^n$ such that $v(t) \leqslant w(t)$ for $t \in [-T, T]$ we define the set of functions

$$S(v, w) = \{u : [-T, T] \to R^n : u(t) = u(t + T) \text{ for } t \in [-T, O]$$
$$u \in C^1([0, T], R^n) \text{ and } v(t) \leqslant u(t) \leqslant w(t)\}.$$

Lemma 1. *Let the following conditions be fulfilled:*

1. *The function* $m \in C^1([-T, T], R^n)$.
2. *The function* $k \in C([0, T] \times [-h, T], [0, \infty))$ *and is bounded.*
3. *The following relations hold*

$$\dot{m}(t) \leqslant -Mm(t) - N \int_{t-h}^{t} k(t, s)m(s)\mathrm{d}s \text{ for } t \in [0, T]$$

$$\dot{m}(t) = m(t + T) \qquad \text{for } t \in [-T, 0]$$

where $M, N = \text{const} > 0$ *are such that*

$$2NTk_0(\mathrm{e}^{Mh} - 1) \leqslant M \tag{4}$$

and $k_0 = \max\limits_{[0,T]\times[-h,T]} k(t, s)$.

Then $m(t) \leqslant 0$ *for* $t \in [-T, T]$.

Proof. Consider the function $g(t) = m(t)\mathrm{e}^{Mt}$. The function $g(t)$ satisfies the relations

$$\dot{g}(t) \leqslant -N \int_{t-h}^{t} k(t, s)\mathrm{e}^{M(t-s)}g(s)\mathrm{d}s \text{ for } t \in [0, T] \tag{5}$$

$$g(t)\mathrm{e}^{MT} = g(t + T) \text{ for } t \in [-T, 0]. \tag{6}$$

We shall show that $g(t) \leqslant 0$ for $t \in [-T, T]$. Suppose that this is not true, i.e. there exists a point $\xi \in [-T, T]$ such that $g(\xi) > 0$. By equality (6) without loss of generality we can assume that $\xi \in [0, T]$. Consider the following two cases:

Case 1. Let $g(t) \geqslant 0$ for $t \in [0, T]$. Then from equality (6) it follows that $g(t) \geqslant 0$ for $t \in [-T, T]$. From inequality (5) and the properties of the function $k(t, s)$ we obtain the inequality $\dot{g}(t) \leqslant 0$ for $t \in [0, T]$, i.e. the function $g(t)$ is nonincreasing in the interval $[0, T]$. Hence

$$g(0) \geqslant g(T) = \mathrm{e}^{MT}g(0). \tag{7}$$

Inequality (7) is fulfilled if and only if $g(0) = 0$. But in this case from equality (6) it follows that $g(T) = 0$. Hence $g(T) = 0 < g(\xi)$ which contradicts the fact that the function is monotone in the interval $[0, T]$. The contradiction obtained implies that the assumption is not true.

Case 2. Let a point $\eta \in [0, T]$ exist such that $g(\eta) < 0$. Introduce the notation $\inf\{g(t) : t \in [0, T]\} = -\lambda$ where $\lambda = \text{const} > 0$. From the continuity of the function $g(t)$ it follows that there exists a point $\zeta \in [0, T]$ such that $g(\zeta) = -\lambda$.

Case 2.1. Let $g(T) \geqslant 0$. Then $\zeta \neq T$ and by the finite increment theorem there exists a point $t_0 \in (0, T]$ such that

$$\dot{g}(t_0) = \frac{g(T) - g(t_0)}{T - t_0} > \frac{\lambda}{T}. \tag{8}$$

For $t_0 > h$ from inequality (5) we obtain

$$\dot{g}(t_0) \leqslant -N \int_{t_0-h}^{t_0} k(t_0, s)\mathrm{e}^{M(t_0-s)}g(s)\mathrm{d}s \leqslant$$

$$\leqslant N\lambda k_0 \int_{t_0-h}^{t_0} \mathrm{e}^{M(t_0-s)}\mathrm{d}s \leqslant \frac{N}{M}\lambda k_0(\mathrm{e}^{Mh} - 1). \tag{9}$$

For $t_0 \leqslant h$ from (5) and (6) it follows that

$$\dot{g}(t_0) \leqslant -N \int_{t_0-h}^{0} k(t_0, s)\mathrm{e}^{M(t_0-s)}g(s + T)\mathrm{e}^{-MT}\mathrm{d}s -$$

$$-N \int_{0}^{t_0} k(t_0, s)\mathrm{e}^{M(t_0-s)}g(s)\mathrm{d}s \leqslant$$

$$\leqslant \frac{N}{M}k_0\lambda[\mathrm{e}^{M(h-T)} - \mathrm{e}^{M(t_0-T)} + \mathrm{e}^{Mt_0} - 1] \leqslant \tag{10}$$

$$\leqslant 2\frac{N}{M}\lambda k_0(\mathrm{e}^{Mh} - 1).$$

Inequalities (8)—(10) contradict inequality (4).

Case 2.2. Let $g(T) < 0$. By the assumption there exists a point t_1 such that $g(t_1) = 0$. The finite increment theorem implies the existence of a point $t_0 \in [0, T]$ for which the following relation holds

$$\dot{g}(t_0) = \frac{g(t_1) - g(\zeta)}{t_1 - \zeta} = \frac{\lambda}{t_1 - \zeta} > \frac{\lambda}{T}.$$

By arguments analogous to those in the proof of Case 2.1 we get to a contradiction.

Hence, $g(t) \leqslant 0$ for $t \in [-T, T]$ which shows that $m(t) \leqslant 0$ for $t \in [-T, T]$.

This completes the proof of Lemma 1.

By means of Lemma 1 we shall justify the monotone-iterative method for construction of quasisolutions of periodic problem (1).

THEOREM 1. *Let the following conditions hold*

1. *The functions $v, w : [-T, T] \to R^n$ are a couple of lower and upper quasisolutions of periodic problem (1) and $v(t) \leqslant w(t)$ for $t \in [-T, T]$.*

2. *The function $f \in C([0, T) \times R^n \times R^n, R^n)$ is monotone nondecreasing in $[u]_{p_j}$ and $[Gu]_{p_j}$ and monotone nonincreasing in $[u]_{q_j}$ and $[Gu]_{q_j}$ and for $v(t) \leqslant y \leqslant x \leqslant w(t)$ satisfies the inequality*

$$f_j(t, x_j, [x]_{p_j}, [x_{q_j}], G_j x, [Gx]_{p_j} [Gx]_{q_j}) -$$

$$- f_j(t, y_j, [x]_{p_j}, [x]_{q_j}, G_j y, [Gx]_{p_j}, [Gx]_{q_j}) \geqslant$$

$$\geqslant -M_j(x_j - y_j) - N_j(G_j x - G_j y), \quad j = \overline{1, n},$$

where M_j, $N_j = \text{const} > 0$

3. *The functions $k_j \in C([0, T] \times [-h, T], [0, \infty))$, $j = \overline{1, n}$ are bounded.*

4. *The inequalities*

$$2 N_j T K_j (e^{M_j h} - 1) \leqslant M_j, \quad j = \overline{1, n},$$

hold where $K_j = \max\limits_{[0,T] \times [-h,T[} k_j(t, s)$.

Then there exist monotone sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ which are uniformly convergent for $t \in [-T, T]$ and their limits $\bar{v}(t) = \lim\limits_{k\to\infty} v^{(k)}(t)$ and $\bar{w}(t) = \lim\limits_{k\to\infty} w^{(k)}(t)$ are a couple of minimal and maximal quasisolutions of periodic problem (1). Moreover, if $u(t)$ is any solution of periodic problem (1) such that $v(t) \leqslant u(t) \leqslant w(t)$ for $t \in [-T, T]$, then the inequalities $\bar{v}(t) \leqslant u(t) \leqslant \bar{w}(t)$ hold for $t \in [-T, T]$.

Proof. Fix the functions $\eta, \mu \in S(v, w)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. Consider the periodic problems

(11)
$$\dot{x}_j + M_j x_j + N_j G_j x = \sigma_j(t \quad \text{for} \quad t \in [0, T]$$

$$x_j(t) = x_j(t + T) \quad \text{for} \quad t \in [-T, 0]$$

where

$$\sigma_j(t) = f_j(t, \eta_j, [\eta]_{p_j}, [\mu]_{q_j}, G_j \eta, [G\eta]_{p_j}, [G\mu]_{q_j}) + M_j \eta_j(t) + N_j G_j \eta(t),$$

$$j = \overline{1, n}.$$

By the conditions of Theorem 1 problem (11) has a solution. From Lemma 1 it follows that the solution of problem (11) is unique.

Define the mapping $A : S(v, w) \times S(v, w) \to C^1([-T, T], R^n)$ by the equality $A(\eta, \mu) = x$, where $x = (x_1, x_2, \ldots, x_n)$ is the unique solution of periodic problem (11) for the couple of functions η, μ.

We shall prove that $v \leqslant A(v, w)$. Introduce the notations $v^{(1)} = A(v, w)$ and $p = v - v^{(1)}$. Then the following inequalities hold

(12)
$$p_j \leqslant -M_j p_j - N_j \int_{t-h}^{t} k_j(t, s) p_j(s) ds \quad \text{for} \quad t \in [0, T]$$

$$p_j(t) = p_j(t + T) \quad \text{for} \quad t \in [-T, 0], \; j = \overline{1, n}.$$

From Lemma 1 it follows that the functions $p_j(t)$ are nonpositive, i.e. $v \leqslant v^{(1)} = A(v, w)$. In an analogous way it is proved that $w \geqslant A(w, v)$.

Let $\eta, \mu \in S(v, w)$ be such that $\eta(t) \leqslant \mu(t)$ for $t \in [-T, T]$. Set $x^{(1)} = A(v, w)$, $x^{(2)} = A(w, v)$ and $p = x^{(1)} - x^{(2)}$. By Lemma 1 the functions $p_j(t)$, $j = \overline{1, n}$ are nonpositive for $t \in [-T, T]$, i.e. the inequality $A(v, w) \leqslant A(w, v)$ holds.

Define the sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ by the equalities

$$v^{(0)} = v, \qquad w^{(0)} = w$$

$$v^{(k+1)} = A(v^{(k)}, w^{(k)}), \quad w^{(k+1)} = A(w^{(k)}, v^{(k)}).$$

The functions $v^{(k)}(t)$ and $w^{(k)}(t)$, $k \geqslant 0$ for $t \in [-T, T]$ satisfy the inequalities

$$v^{(0)}(t) \leqslant v^{(1)}(t) \leqslant \ldots \leqslant v^{(k)}(t) \leqslant \ldots \leqslant w^{(k)}(t) \leqslant \ldots \leqslant w^{(0)}(t).$$

Hence the sequences $\{v^{(k)}(t)\}_0^\infty$ and $w^{(k)}(t)\}_0^\infty$ are uniformly convergent for $t \in [-T, T]$. Introduce the notations $\bar{v}(t) = \lim\limits_{k\to\infty} v^{(k)}(t)$ and $\bar{w}(t) = \lim\limits_{k\to\infty} w^{(k)}(t)$. We shall show that $\bar{v}$, $\bar{w}$ are a couple of minimal and maximal quasisolutions of periodic problem (1). From inequalities (13) it follows that there exists an integer k such that $v^{(k-1)}(t) \leqslant u_1(t) \leqslant w^{(k-1)}(t)$ and $v^{(k-1)}(t) \leqslant u_2(t) \leqslant w^{(k-1)}(t)$ for $t \in [-T, T]$ where $u_1, u_2 \in S(v, w)$ are a couple of quasisolutions of (1).

Introduce the notations $p_j(t) = v_j^{(k)}(t) - u_{1j}(t)$. From condition 2 of the

Theorem 1 if follows the inequalities

$$\dot{p}_j(t) = f_j(t, v_j^{(k-1)}, [v^{(k-1)}]_{p_j}, [w^{(k-1)}]_{q_j}, G_j v^{(k-1)}, [Gv^{(k-1)}]_{p_j}, [Gw^{(k-1)}]_{q_j}) -$$
$$- f_j(t, u_{1j}, [u_1]_{p_j}, [u_2]_{q_j}, G_j u_1, [Gu_1]_{p_j}, [Gu_2]_{q_j}) -$$
$$- M_j(v_j^{(k)} - v_j^{(k-1)}) - N_j(G_j v^{(k)} - G_j v^{(k-1)}) \leqslant$$
$$\leqslant - M_j p_j(t) - N_j G_j p \qquad \text{for } t \in [0, T]$$
$$p_j(t) = p_j(t+T) \qquad \text{for } t \in [-T, 0].$$

By Lemma 1 the functions $p_j(t)$, $j = \overline{1, n}$ are nonpositive, i.e. $v^{(k)}(t) \leqslant u_1(t)$ for $t \in [-T, T]$.

In an analogous way it is proved that the inequalities

$$u_1(t) \leqslant w^{(k)}(t) \quad \text{and} \quad v^{(k)}(t) \leqslant u_2(t) \leqslant w^{(k)}(t)$$

hold for $t \in [-T, T]$ which shows that the couple $\bar{v}$, $\bar{w}$ is a couple of minimal and maximal quasisolutions of periodic problem (1).

Let $x(t)$ be any solution of problem (1) such that $v(t) \leqslant x(t) \leqslant w(t)$. Consider the couple (x, x) which is a couple of quasisolutions of (1). From the fact that the couple $(\bar{v}, \bar{w})$ is a couple of minimal and maximal quasisolutions of (1) it follows that the inequalities $\bar{v}(t) \leqslant x(t) \leqslant \bar{w}(t)$ hold for $t \in [-T, T]$.

This completes the proof of Theorem 1.

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