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AN EXACT ESTIMATE FOR THE APPROXIMATION OF THE FUNCTION Msgn$(A-X)$ WITH BERNSTEIN POLYNOMIALS IN HAUSDORFF METRIC

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1. Notations and definitions. F_Δ — the set of all segment bounded functions, defined on the interval $\Delta : f \in F_\Delta$ represents Δ in the set of all intervals.

For every $f \in F_\Delta$, $\delta > 0$ we define

$$I(f;\ x) = \lim_{\delta \to 0} I(\delta,\ f;\ x);$$

$$S(f,\ x) = \lim_{\delta \to 0} S(\delta,\ f;\ x),$$

where

$$I(\delta,\ f;\ x) = \inf\{y : y \in f(t),\ \ t \in [x-\delta,\ x+\delta] \cap \Delta\};$$

$$S(\delta,\ f;\ x) = \sup\{y : y \in f(t),\ \ t \in [x-\delta,\ x+\delta] \cap \Delta\}.$$

The completed graph of $f \in F_\Delta$ is the segment function

$$\bar{f}(x) = [I(f;\ x),\ S(f;\ x)].$$

We define the Hausdorff distance between $f,\ g \in F_\Delta$ according to ([5], pp. 34), as follows

$$\tau([0,\ 1];\ f,\ g) = \max\{\sup_{\mathscr{A} \in \bar{f}} \inf_{B \in \bar{g}} \rho(A,\ B),\ \sup_{A \in \bar{g}} \inf_{B \in \bar{f}} \rho(A,\ B)\},$$

where

$$\rho(A(x,\ y),\ B(\xi,\ \eta)) = \max\{|x-\xi|,\ |y-\eta|\}.$$

The Hausdorff distance between $f,\ g \in F_\Delta$ in the point $x_0 \in \Delta$ we denote

$$\tau(\Delta;\ f(x_0) - g(x_0)) =$$
$$= \max\{\min_{x \in \Delta} \max[|x - x_0|,\ |g(x_0) - f(x)|],$$
$$\min_{x \in \Delta} \max[|x - x_0|,\ |g(x) - f(x_0)|]\}.$$

2. Main results

THEOREM 1. *Let $\sigma \in F_{[0,1]}$ be defined as follows*

$$\sigma(x) = \begin{cases} M, & x \in [0, x_0); \\ 0, & x = x_0; \\ -M, & x \in (x_0, 1], \end{cases}$$

where $0 < x_0 \leqslant 1/2$, $M > 0$.

Then

(1) $$\lim_{n\to\infty} [n^{-1} \ln n]^{-1/2} \tau([0,1]; B_n(\sigma), \sigma) = [x_0(1-x_0)]^{1/2}$$

holds, where

$$B_n(\sigma; x) = \sum_{\nu=0}^{n} \sigma\left(\frac{\nu}{n}\right) p_{n,\nu}(x),$$

$$p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

is the Bernstein polynomial.

THEOREM 2. *Let $\sigma_0 \in F_\Delta$ be defined as follows*

$$\sigma_0(x) = \begin{cases} 0, & x = 0; \\ -M, & x \in (0, 1]. \end{cases}$$

Then for sufficiently large n

(2) $$\tau([0, 1]; B_n(\sigma_0), \sigma_0) \asymp \frac{\ln n}{n}$$

holds, where

$$B_n(\sigma_0; x) = \sum_{\nu=0}^{n} \sigma_0\left(\frac{\nu}{n}\right) p_{n,\nu}(x),$$

$$p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

is the Bernstein polynomial.

3. Proof of Theorem 1. We shall prove the statement in two steps.

1. In the first step we shall prove that for sufficiently large n the inequality

(3) $$\tau([0,1]; B_n(\sigma), \sigma) \geqslant [x_0(1-x_0)n^{-1}\ln n]^{1/2}$$

holds.

In view of the properties of Bernstein polynomials and the definition of the Hausdorff distance we can say that (3) is valid, if we prove that the inequality

$$B_n(\sigma; x_0^*) > -M + \frac{1}{\lambda} \text{Const.}\ M n^{-\lambda/2}$$

holds, where $x_0^* = x_0 + \lambda\delta(x_0; n)$, $0 < \lambda < 1$ is an arbitrary point on the interval $(x_0, x_0 + \delta(x_0; n))$, $\delta(x_0; n) = [x_0(1-x_0)n^{-1}\ln n]^{1/2}$.

Really, after some transformations we obtain

$$B_n(\sigma; x_0^*) = \sum_{\nu=0}^{n} \sigma\left(\frac{\nu}{n}\right) p_{n,\nu}(x_0^*) =$$

(4) $$= \sum_{\left|x_0^*-\frac{\nu}{n}\right| \leqslant \lambda\delta(x_0;n)} \sigma\left(\frac{\nu}{n}\right) p_{n,\nu}(x_0^*) + \sum_{\left|x_0^*-\frac{\nu}{n}\right| > \lambda\delta(x_0;n)} \sigma\left(\frac{\nu}{n}\right) p_{n,\nu}(x_0^*)$$

$$= -M\Big[1 - \sum_{\left|x_0^*-\frac{\nu}{n}\right| > \lambda\delta(x_0;n)} p_{n,\nu}(x_0^*)\Big] + \sum_{\left|x_0^*-\frac{\nu}{n}\right| > \lambda\delta(x_0;n)} \sigma\left(\frac{\nu}{n}\right) p_{n,\nu}(x_0^*)$$

$$= -M + \sum_{\left|x_0^*-\frac{\nu}{n}\right| > \lambda\delta(x_0;n)} \left[M + \sigma\left(\frac{\nu}{n}\right)\right] p_{n,\nu}(x_0^*)$$

$$= -M + 2M \sum_{\frac{\nu}{n} < x_0^* - \lambda\delta(x_0;n)} p_{n,\nu}(x_0^*).$$

Now we take into account, that if $\delta(n) \leqslant n^{-\gamma}\left(\gamma > \frac{1}{3}\right)$, $n\delta(n) \to \infty$, for sufficiently large n

$$\sum_{\left|x-\frac{\nu}{n}\right| \leqslant \delta(n)} p_{n,\nu}(x) \sim \frac{2}{\sqrt{\pi}} \int_0^{\omega(n)} \exp(-v^2) dv,$$

is valid, where $\omega(n) = n^{1/2}\delta(n)[2x(1-x)]^{-1/2}$ ([4], pp. 18).

If is clear that if $\lim_{n\to\infty} (n^{1/2}\delta(n))^{-1} = 0$, then

$$\lim_{n\to\infty} \sum_{\left|x-\frac{\nu}{n}\right| \leqslant \delta(n)} p_{n,\nu}(x) = 1$$

Hence for sufficiently large n the asymptotic equality

(5) $$\sum_{\left|x-\frac{\nu}{n}\right| > \delta(n)} p_{n,\nu}(x) \sim \frac{2}{\sqrt{\pi}} \int_{\omega(n)}^{\infty} \exp(-v^2) dv$$

holds. Then, using (4) and (5) we have

(6) $$B_n(\sigma; x_0^*) \sim -M + \frac{2M}{\sqrt{\pi}} \int_{\omega(n)}^{\infty} \exp(-v^2) dv,$$

where

$$\omega(\lambda;\ n) = \lambda\delta(x_0;\ n)[2n^{-1}x_0^*(1-x_0^*)]^{-1/2} = \left[\frac{\lambda^2}{2}\cdot\frac{x_0(1-x_0)}{x_0^*(1-x_0^*)}\cdot\ln n\right]^{1/2}.$$

Now we observe two cases

a) If $0 < x_0 < 1/2$, then for sufficiently large n the inequality $x_0 < x_0^* \leqslant 1/2$ is true and

$$[x_0(1-x_0)]\cdot[x_0^*(1-x_0^*)]^{-1} < 1$$

gets. Then we can imply that

$$\omega(\lambda;n) \leqslant [\lambda^2 \ln n/2]^{1/2}.$$

b) If $x_0 = 1/2$, we have

$$\frac{x_0(1-x_0)}{x_0^*(1-x_0^*)} = \{[1+\lambda(n^{-1}\ln n)^{1/2}][1-\lambda(n^{-1}\ln n)^{1/2}]\}^{-1} = 1+\varepsilon(n),$$

where $\lim_{n\to\infty} \varepsilon(n) = 0$. In this case

$$\omega(\lambda;n) = [\lambda^2(1+\varepsilon(n))\ln n/2]^{1/2}$$

holds. But let us note that $0 < \lambda < 1$ and for sufficiently large n, $\varepsilon(n)$ can be small enough such that $\lambda(1+\varepsilon(n)) < 1$.

Therefore we show that if $0 < x_0 \leqslant 1/2$, then for sufficiently large n

$$(7)\qquad B_n(\sigma;x_0^*) > -M + \frac{2M}{\sqrt{\pi}}\int\limits_{[\lambda^2(1+\varepsilon(n))\ln n/2]^{1/2}}^{\infty} \exp(-v^2)\mathrm{d}v$$

is true, where $0 < \lambda < 1$, $\varepsilon(n)$ is small enough such that $\lambda(1+\varepsilon(n)) < 1$.

On the other hand it is proved the inequality ([6], pp. 166)

$$(8)\qquad \int\limits_z^{\infty} \exp(-t^2)\mathrm{d}t > \frac{1}{2z}\left(1-\frac{1}{2z^2}\right)\exp(-z^2).$$

Then from (7) and (8) follow

$$(9)\qquad B_n(\sigma;x_0^*) > -M + 2M[2\pi\lambda^2(1+\varepsilon(n))\ln n]^{-1/2}\cdot$$

$$\cdot[1-(\lambda^2(1+\varepsilon(n))\ln n)^{-1}].n^{-\lambda^2(1+\varepsilon(n))/2}$$

$$> -M + \frac{2M}{4\lambda\sqrt{\pi}}\left\{\frac{n^{\lambda[1-\lambda(1+\varepsilon(n))]}}{\ln n}\right\}^{1/2}\cdot n^{-\lambda/2}$$

$$> -M + \frac{2MC_2}{4\lambda\sqrt{\pi}}\cdot n^{-\lambda/2} > -M + \frac{C_3M}{\lambda}\cdot n^{-\lambda/2},$$

where $x_0^* = x_0 + \lambda\delta(x_0;n)$ is an arbitrary point in $(x_0,\ x_0+\delta(x_0;\ n))$.

Further in view of the definition of Hausdorff distance, it is not hard to see that (3) is valid, as $\lim_{n\to\infty} n^{\lambda/2}(n^{-1}\ln n)^{1/2} = 0$, when $0 < \lambda < 1$.

2) In the second step we shall prove, for sufficiently large n the inequality

$$(10)\qquad \tau[(0,\ 1];\ B_n(\sigma),\ \sigma)) \leqslant [x_0(1-x_0)n^{-1}\ln n]^{1/2}$$

holds.

In view of the definition of the Hausdorff distance the statement shall be valid, if we show that the uniform distance between Bernstein polynomials and the function σ in the intervals

$$[0,\ x_0-\delta(x_0;\ n))\ \text{and}\ (x_0+\delta(x_0;\ n),\ 1]$$

is smaller than $\delta(x_0;\ n) = [x_0(1-x_0)n^{-1}\ln n]^{1/2}$.

a) Let $x_{\mu_1} = x_0 - \mu_1\delta(x_0;n)$, $\mu_1 \geqslant 1$ be an arbitrary point in the interval $[0,\ x_0-\delta(x_0;\ n))$. By the Bernstein polynomial we have

$$(11)\qquad B_n(\sigma;x_{\mu_1}) = M - \sum_{\left|x_{\mu_1}-\frac{\nu}{n}\right| > \mu_1\delta(x_0;n)} \left[M - \sigma\left(\frac{\nu}{n}\right)\right]p_{n,\nu}(x_{\mu_1}) =$$

$$= M - 2M \sum_{\frac{\nu}{n} > x_{\mu_1}+\mu_1\delta(x_0;n)} p_{n,\nu}(x_{\mu_1})$$

$$\sim M - \frac{2M}{\sqrt{\pi}}\int\limits_{\omega_1(n)}^{\infty}\exp(-v^2)\mathrm{d}v,$$

where

$$\omega_1(n) = \mu_1[x_0(1-x_0)n^{-1}\ln n]^{1/2}\cdot n^{1/2}[2x_{\mu_1}(1-x_{\mu_1})]^{-1/2}$$

$$= \mu_1\left[\frac{x_0(1-x_0)}{2x_{\mu_1}(1-x_{\mu_1})}\cdot\ln n\right]^{1/2} > [\ln n/2]^{1/2},$$

as $\mu_1 \geqslant 1$ and $x_{\mu_1} < x_0 \leqslant 1/2$.

But it is known ([6], pp. 166) that

$$(12)\qquad \int\limits_z^{\infty}\exp(-t^2)\mathrm{d}t < \frac{1}{2z}\exp(-z^2)$$

holds. Hence (11) and (12) yield

$$(13)\qquad B_n(\sigma;x_{\mu_1}) \geqslant M - \frac{2M}{\sqrt{\pi}}\int\limits_{(\ln n/2)^{1/2}}^{\infty}\exp(-v^2)\mathrm{d}v$$

$$> M - \frac{2M}{(2\pi n\ln n)^{1/2}}.$$

b) Let $x_{\mu_2} = x_0 + \mu_2\delta(x_0; n)$, $\mu_2 > 1$ be an arbitrary point in $(x_0 + \delta(x_0; n), 1]$. In this case we obtain:

$$B_n(\sigma; x_{\mu_2}) = -M + \sum_{\left|x_{\mu_2} - \frac{\nu}{n}\right| > \mu_2\delta(x_0;n)} \left[M + \sigma\left(\frac{\nu}{n}\right)\right] p_{n,\nu}(x_{\mu_2})$$

$$= -M + 2M \sum_{\frac{\nu}{n} < x_{\mu_2} + \mu_2\delta(x_0;n)} p_{n,\nu}(x_{\mu_2})$$

$$\sim -M + \frac{2M}{\sqrt{\pi}} \int_{\omega_2(n)}^{\infty} \exp(-v^2)dv,$$

where

$$\omega_2(n) = \mu_2[x_0(1-x_0)n^{-1}\ln n]^{1/2} . n^{1/2} \cdot [2x_{\mu_2}(1 - x_{\mu_2})]^{-1/2}$$

$$= \mu_2\left[\frac{x_0(1-x_0)}{2x_{\mu_2}(1-x_{\mu_2})} \cdot \ln n\right]^{1/2}.$$

i) If $x_0 = 1/2$, then $\omega_2(n) > (\ln n/2)^{1/2}$.
ii) If $0 < x_0 < 1/2$, then

$$\frac{x_0(1-x_0)}{x_{\mu_2}(1-x_{\mu_2})} = \left[\left(1 + \mu_2\sqrt{\frac{1-x_0}{x_0} \cdot \frac{\ln n}{n}}\right)\left(1 - \mu_2\sqrt{\frac{x_0}{1-x_0} \cdot \frac{\ln n}{n}}\right)\right]^{-1}$$

$$> \left[1 + \mu_2\frac{1-2x_0}{\sqrt{x_0(1-x_0)}} \cdot \sqrt{\frac{\ln n}{n}}\right]^{-1} = 1 - \varepsilon(n)$$

holds, where $\lim_{n\to\infty} \varepsilon(n) = 0$. Since $\mu_2 > 1$, for sufficiently large n is true $\mu_2(1 - \varepsilon(n)) \geqslant 1$ and we can conclude

$$\omega_2(n) \geqslant (\ln n/2)^{1/2}.$$

Now using (12) we have

$$B_n(\sigma; x_{\mu_2}) < -M + \frac{2M}{\sqrt{\pi}} \int_{(\ln n/2)^{1/2}}^{\infty} \exp(-v^2)dv \tag{14}$$

$$< -M + \frac{2M}{(2\pi n \ln n)^{1/2}}.$$

Further (13), (14) and the definition of Hausdorff distance yield (10). The theorem is proved.

IV. *Proof. of Theorem* 2. At first we shall prove that for sufficiently large n the inequality

$$\tau([0,1]; B_n(\sigma_0), \sigma_0) \leqslant 2\frac{\ln n}{n} \tag{15}$$

holds.

In view of the definition of Hausdorff distance, (15) will be valid, if for every $x \in \left(2\frac{\ln n}{n}, 1\right]$

$$B_n(\sigma_0; x) \leqslant -M + 2\frac{\ln n}{n}$$

gets.

Really, using the asymptotic equation

$$\sum_{\left|x - \frac{\nu}{n}\right| \leqslant \delta(n)} p_{n,\nu}(x) \sim \frac{2}{\sqrt{\pi}} \int_0^{\omega(n)} \exp(-v^2)dv,$$

where

$$\omega(n) = n^{1/2}\delta(n)[2x(1-x)]^{-1/2}$$

for sufficiently large n we have

$$B_n(\sigma_0; x) = \sum_{\left|x - \frac{\nu}{n}\right| \leqslant 2\delta(x;n)} \sigma_0\left(\frac{\nu}{n}\right) p_{n,\nu}(x) \tag{16}$$

$$+ \sum_{\left|x - \frac{\nu}{n}\right| > 2\delta(x;n)} \sigma_0\left(\frac{\nu}{n}\right) p_{n,\nu}(x)$$

$$= -M + \sum_{\left|x - \frac{\nu}{n}\right| > 2\delta(x_0;n)} \left[M + \sigma_0\left(\frac{\nu}{n}\right)\right] p_{n,\nu}(x)$$

$$\leqslant -M + 2M \sum_{\left|x - \frac{\nu}{n}\right| > 2\delta(x_0;n)} p_{n,\nu}(x),$$

where

$$\delta(x; n) = [x(1-x)n^{-1}\ln n]^{1/2}.$$

But it is known, if $0 \leqslant x \leqslant 1$ and $0 \leqslant z \leqslant \frac{3}{2}[nx(1-x)]^{1/2}$ the inequality

$$\sum_{\left|x - \frac{\nu}{n}\right| \geqslant 2z[x(1-x)n^{-1}]^{1/2}} p_{n,\nu}(x) \leqslant 2\exp(-z^2) \tag{17}$$

holds. ([4], pp. 20)

Hence (16) and (17) yield

$$B_n(\sigma_0; x)) \leqslant -M + 2Mn^{-1}$$

which proves (15).

Now we shall prove that for sufficiently large n the inequality

$$\tau([0, 1]; B_n(\sigma_0), \sigma_0) \geq \frac{\ln n}{n} \tag{18}$$

gets.

Using the properties of Bernstein polynomials and the definition of Hausdorff distance we can say (18) is true, if the inequality

$$B_n(\sigma_0; x_\lambda) > -M + Mn^{-\lambda}$$

holds, where $x_\lambda = \lambda n^{-1} \ln n$, $0 < \lambda < 1$.

We calculate

$$\begin{aligned} B_n(\sigma_0; x_\lambda) &= -M + \sum_{\left|x_\lambda - \frac{\nu}{n}\right| > \lambda\delta(x_0; n)} \left[M + \sigma_0\left(\frac{\nu}{n}\right)\right] p_{n,\nu}(x_\lambda) \\ &= -M + Mp_{n,0}(x_\lambda) \\ &= -M + M(1 - \lambda n^{-1} \ln n)^n. \end{aligned}$$

Consequently

$$B_n(\sigma_0; x_\lambda) > -M + M \exp(-\lambda \ln n)$$

or

$$B_n(\sigma_0; x_\lambda) > -M + Mn^{-\lambda},$$

for every $x_\lambda = \lambda n^{-1} \ln n$, $0 < \lambda < 1$.

Now we take into account that

$$n^{-\lambda} \geq n^{-1} \ln n,$$

when $0 < \lambda < 1$ and prove (18).

Finally, it is evident (15) and (18) yield (2).

The proof is complete.

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