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# ON ABSOLUTE SUMMABILITY FACTORS

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In the present paper, the hypotheses of a theorem on absolute summability factors of infinite series have been weakened.

**1.** DEFINITION. Let $\sum a_n$ be an infinite series with partial sums $s_n$ and let $(p_n)$ be a sequence of positive real numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \text{ as } n \to \infty, \quad (P_{-1} = p_{-1} = 0).$$

The sequence-to-sequence transformation :

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (P_n > 0)]$$

defines the sequence $(t_n)$ of $(\bar{N}, p_n)$ mean of the sequence $(s_n)$, generated the sequence of coefficients $(p_n)$. The series $\sum a_n$ is called summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if (see Bor[1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} | t_n - t_{n-1} |^k < \infty.$$

**2.** Bor[1] proved the following theorem in 1987.

THEOREM A. *Let $(x_n)$ be a positive non-decreasing sequence and there be sequences $(\beta_n)$ and $(\lambda_n)$ such that*

$$| \Delta\lambda_n | \leqslant \beta_n \tag{2.1}$$

$$\beta_n \to 0 \quad \text{as } n \to \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} n | \Delta\beta_n | x_n < \infty \tag{2.3}$$

$$| \lambda_n | x_n = O(1) \tag{2.4}$$

*and, for $k \geqslant 1$,*

$$\sum_{v=1}^{n} \frac{1}{v} | s_v |^k = O(x_n) \text{ as } n \to \infty. \tag{2.5}$$

*Suppose further the sequence $(p_n)$ is such that*

$$P_n = 0(np_n) \tag{2.6}$$

$$P_n \Delta p_n = 0(p_n p_{n+1}). \tag{2.7}$$

Then the series $\sum_{n=1}^{\infty} a_n \frac{p_n \lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$ $k \geqslant 1$.

**3.** The object of this paper is, by weakening conditions, to obtain a more general theorem than the one due to Bor[1]. In what follows we shall prove the following theorem.

THEOREM B. *Let $k \geqslant 1$ and let $(x_n)$ be a positive non-decreasing sequence. Under the assumptions (2.1), (2.2), (2.5) and (2.6) of theorem A if the sequences $(\beta_n)$, $(\lambda_n)$ and $(p_n)$ satisfy*

$$\sum_{n=1}^{\infty} x_n |\Delta(|\lambda_n|^k)| < \infty \tag{3.1}$$

$$\sum_{n=1}^{\infty} n^k x_n (\beta_n + \beta_{n+1})^{k-1} |\Delta \beta_n| < \infty \tag{3.2}$$

$$x_n |\lambda_n|^k = 0(1) \tag{3.3}$$

*and*

$$\Delta\left(\frac{P_n}{np_n}\right) = 0\left(\frac{1}{n}\right) \quad as\ n \to \infty. \tag{3.4}$$

*Then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$.*

It may be remarked one of the conditions (3.1) and (3.2) does not imply to the other under the conditions of theorem A.

On the other hand the hypotheses of theorem A imply the hypotheses of theorem B, but the converse of this implication need not be true. This can be shown as follows. By observation of Mishra [2], the conditions (2.2) and (2.3) imply that

$$nx_n \beta_n = 0(1) \tag{3.5}$$

and

$$\sum_{n=1}^{\infty} \beta_n x_n < \infty. \tag{3.6}$$

By (3.5), we can also write

$$n(\beta_n + \beta_{n+1}) = 0(1).$$

Hence, we have

$$\sum_{n=1}^{\infty} n^k x_n (\beta_n + \beta_{n+1})^{k-1} |\Delta \beta_n| = \sum_{n=1}^{\infty} nx_n \{n(\beta_n + \beta_{n+1})\}^{k-1} |\Delta \beta_n| =$$

$$= 0(1) \sum_{n=1}^{\infty} nx_n |\Delta \beta_n| = 0(1),$$



and considering that

$$\Delta |\lambda_n|^k = |\lambda_n|^k - |\lambda_{n+1}|^k = k \xi_n^{k-1} (|\lambda_n| - |\lambda_{n+1}|) \tag{3.7}$$

where $\xi_n$ lies between $|\lambda_n|$ and $|\lambda_{n+1}|$, we have

$$\sum_{n=1}^{\infty} x_n |\Delta(|\lambda_n|^k)| = k \sum_{n=1}^{\infty} x_n \xi_n^{k-1} |\Delta(|\lambda_n|)| =$$

$$= 0(1) \sum_{n=1}^{\infty} x_n \beta_n = 0(1), \quad \text{by (3.6)}.$$

Hence, (2.2) and (2.3) $\Rightarrow$ (3.1) and (3.2). In addition, it is clear that (2.4)$\Rightarrow$(3.3) and, that (2.6) and (2.7)$\Rightarrow$(3.4) (see Mishra and Srivastava [3]). To show the converse, it is sufficient to take that $x_n = \log(n+1)$, $\lambda_n = x_n^{-1}$, $\Delta\lambda_n = \beta_n$ and $k > 1$.

**4.** For the proof of the theorem we require the following Lemma.

LEMMA. *If the sequence $(\beta_n)$ satisfies the conditions (2.2) and (3.2) then, for $k \geqslant 1$,*

$$x_n (n\beta_n)^k = 0(1) \tag{4.1}$$

*and*

$$\sum_{v=1}^{n} v^{k-1} x_v \beta_v^k = 0(1) \quad as\ n \to \infty \tag{4.2}$$

*Proof.* Since $\beta_n \to \infty$ as $n \to \infty$ by (2.2), we have

$$x_n (n\beta_n)^k \leqslant x_n n^k \sum_{v=n}^{\infty} |\Delta \beta_v^k| \leqslant \sum_{v=n}^{\infty} v^k x_v (\beta_v + \beta_{v+1})^{k-1} |\Delta \beta_v| = 0(1) \text{ by (3.2)},$$

and

$$\sum_{v=1}^{n} v^{k-1} x_v \beta_v^k = 0(1) \sum_{v=1}^{\infty} v^{k-1} x_v \sum_{i=v}^{\infty} |\Delta \beta_i^k| = 0(1) \sum_{i=1}^{\infty} |\Delta \beta_i^k| \sum_{v=1}^{i} v^{k-1} x_v =$$

$$= 0(1) \sum_{i=1}^{\infty} i^k x_i (\beta_i + \beta_{i+1})^{k-1} |\Delta \beta_i| = 0(1) \quad \text{by (3.2)}.$$

**5.** *Proof of the theorem.* For establishing the theorem we have to prove that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{vp_v}.$$

So,

$$T_n - T_{n-1} = \Delta\left(\frac{1}{P_{n-1}}\right)\sum_{v=0}^{n}\frac{P_{v-1}P_v a_v \lambda_v}{v p_v}, \quad n \geqslant 1.$$

Abel's transformation enables us to get that

$$T_n - T_{n-1} = \Delta\left(\frac{1}{P_{n-1}}\right)\sum_{v=1}^{n-1}\frac{P_v P_{v+1} + \Delta\lambda_v s_v}{(v+1)p_{v+1}} - \Delta\left(\frac{1}{P_{n-1}}\right)\sum_{v=1}^{n-1}\frac{P_v\lambda_v s_v}{v} +$$

$$+\Delta\left(\frac{1}{P_{n-1}}\right)\sum_{v=1}^{n-1}P_v\Delta\left(\frac{P_v}{vp_v}\right)\lambda_v s_v + \frac{\lambda_n}{n}s_n = T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}, \text{ say.}$$

By Minkowki's inequality, it is sufficient to show that $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,i}|^k < \infty$ for $i = 1, 2, 3, 4$.

Now, using (2.6), we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k = 0(1)\sum_{n=2}^{m+1}\left(\frac{1}{P_{n-1}}\right)^{k-1}\Delta\left(\frac{1}{P_{n-1}}\right)\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}\cdot p_v|\Delta\lambda_v||s_v|\right\}^k$$

Hölder's inequality gives us that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k = 0(1)\sum_{n=2}^{m+1}\Delta\left(\frac{1}{P_{n-1}}\right)\times$$

$$\times\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\lambda_v|^k|s_v|^k x\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} =$$

$$= 0(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\Delta\left(\frac{1}{P_{n-1}}\right) =$$

$$= 0(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^{k-1}v^k|\Delta\lambda_v|^k\frac{|s_v|^k}{v} = 0(1)\sum_{v=1}^{m}v^k\beta_v^k\frac{|s_v|^k}{v}$$

by virtues of (2.1) and (2.6).

Replacing $|\lambda_v|$ by $\beta_v$ in equality (3.7), we have that $|\Delta\beta_v^k| = 0(1)(\beta_v + \beta_{v+1})^{k-1}|\Delta\beta_v|$. So,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k = 0(1)\sum_{v=1}^{m-1}v^k x_v(\beta_v + \beta_{v+1})^{k-1}|\Delta\beta_v| +$$

$$+ 0(1)\sum_{v=1}^{m-1}(v+1)^{k-1}x_{v+1}\beta_{v+1}^k + 0(1)x_m(m\beta_m)^k =$$

$= 0(1)$ as $m \to \infty$, by (3.2), (4.1) and (4.2).



Again, by (2.6), we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k = \sum_{n=2}^{m+1}\left(\frac{1}{P_{n-1}}\right)^{k-1}\Delta\left(\frac{1}{P_{n-1}}\right)\left|\sum_{v=1}^{n-1}\frac{P_v}{vp_v}p_v\lambda_v s_v\right|^k =$$

$$= 0(1)\sum_{n=2}^{m+1}\Delta\left(\frac{1}{P_{n-1}}\right)\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k p_v|\lambda_v|^k|s_v|^k x\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} =$$

$$= 0(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k p_v|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\Delta\left(\frac{1}{P_{n-1}}\right) = 0(1)\sum_{v=1}^{m}|\lambda_v|^k\frac{|s_v|^k}{v} =$$

$= 0(1)\sum_{v=1}^{m-1}x_v|\Delta(|\lambda_v|^k)| + 0(1)x_m|\lambda_m|^k = 0(1)$ as $m \to \infty$, by (3.1) and (3.3).

Hence, considering (3.4), as in $T_{n,2}$, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k = \sum_{n=2}^{m+1}\left(\frac{1}{P_{n-1}}\right)^{k-1}\Delta\left(\frac{1}{P_{n-1}}\right)\left|\sum_{v=1}^{n-1}P_v\Delta\left(\frac{P_v}{vp_v}\right)\lambda_v s_v\right|^k$$

$= 0(1)\sum_{n=2}^{m+1}\left(\frac{1}{P_{n-1}}\right)^{k-1}\Delta\left(\frac{1}{P_{n-1}}\right)\left|\sum_{v=1}^{n-1}\frac{P_v}{v}\lambda_v s_v\right|^k = 0(1)$ as $m \to \infty$.

Finally, as in $T_{n,2}$, we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k = \sum_{n=1}^{m}\left(\frac{P_n}{np_n}\right)^{k-1}|\lambda_n|^k\frac{|s_n|^k}{n} =$$

$= 0(1)\sum_{n=1}^{m}|\lambda_n|^k\frac{|s_n|^k}{n} = 0(1)$ as $m \to \infty$,

which completes the proof.

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