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FAMILIES OF THIRD, FOURTH & FIFTH ORDER EXTENDED NYSTRÖM METHODS FOR $Y''' = f(x, y)$

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Abstract. Regarding the existence of special third order non-linear initial value problem, Hairer et al [8] have mentioned in their text book that special equations of the type $Y''' = f(x, y)$, with suitable initial conditions, appear more frequently than the general ones. They are also useful in many physical problems. For these equations the present author has given the theory and derivation of order conditions governing the parameters of the Extended Nyström methods in references [10]. In this paper we want to present the derivations of families of third, fourth & fifth order Extended Nyström methods and their special cases for solving special third order non-linear initial value problems.

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1. Introduction. Keeping in view the remarks made by Hairer et al [18] about the occurrence and utility of the special third order non-linear initial value problem in many physical problems and using the theory for Extended Nyström methods, developed by the present author [10], in this work we want to present families of different order Extended Nyström method (which we denote as EN-method from now onwards) to solve the nonstiff equations of the type $y''' = f(x, y)$ with suitable initial conditions. The general approach, for a given order s, appears as shown below.

For the special third order non-linear initial value problem

$$y''' = f(x, y) \text{ with, } y(x_0) = y_0,\ y'(x_0) = y_0',\ y''(x_0) = y_0'' \tag{1}$$

the sth order EN-method is defined as

$$\begin{aligned} y_1 &= y(x_0 + h) = y_0 + h\alpha_s y_0' + h^2 \nu_s y_0'' + h^3 \sum_{j=1}^{s-1} a_{sj} k_j + T_0(h) \\ y_1' &= y'(x_0 + h) = y_0' + h\beta_s y_0'' + h^2 \sum_{j=1}^{s-1} \bar{a}_{sj} k_j + T_0'(h) \\ y_1'' &= y''(x_0 + h) = y_0'' + h \sum_{j=1}^{s-1} \bar{\bar{a}}_{sj} k_j + T_0''(h) \end{aligned} \tag{2}$$

where

$$k_j = f\left(x_i + \alpha_i h, y_0 + h\alpha_i y_0' + h^2 \nu_i y_0' + h^3 \sum_{j=1}^{s-1} a_{ij}k_j\right)$$

The general procedure adopted in deriving various order EN-method is to get from table (3) of Sarma [10] a system of necessary equations of order conditions governing the parameters of the method defined in (2). Then following Chawala & Sharma [2–4] and using equation (11) (due to simplifying assumption) in proposition 3 of reference [10] (which for readers ready reference is presented here $P(P-1)(P-2)\sum_{l=1}^{k-1} a_{kl}\alpha_l^{P-3} = \alpha_s^P$, $k = 2, 3, \dots s$. ——— (3) for different values of P & k, we supplement to the above system additional relations governing the unknowns of the method (2). Finally solution of this system is obtained for various unknowns in general and also in cases of exception. The derivation of families of third, fourth and fifth order EN-methods are presented in sections 2, 3 & 4.

2. Families of Third order Extended Nyström methods. For the special third order non-linear initial value problem (1), a class of third order (explicit) EN-method may be defined as

$$y_1 = y_0 + h\alpha_3 y_0' + h^2\nu_3 y_0'' + h^3 \sum_{j=1}^{2} a_{3j}k_j + T_0(h)$$

$$y_1' = y_0' + h\beta_3 y_0' + h^2 \sum_{j=1}^{2} \bar{a}_{3j}k_j + T_0'(h) \qquad (2.1)$$

$$y_1'' = y_0'' + h \sum_{j=1}^{2} \bar{\bar{a}}_{3j} k_j + T_0''(h)$$

where $k_i = f(x_i + \alpha_i h, y_0 + h\alpha_i y_0' + h^2\nu_i y_0' + h^3 \sum_{j=1}^{i-1} a_{ij}k_j)$

which involves eleven parameters, namely, a_{21} & α_i, ν_i, a_{3i}, $\bar{a}_{3i}$, $\bar{\bar{a}}_{3i}$ for $i = 1,2$.

Let $T_0(h)$, $T_0'(h)$ and $T_0''(h)$ be each of order $O(h^4)$ then from table 3 of Sarma [10] there result for third order explicit EN-method defined by (2.1), a system of seven equations involving ten parameters, given by

$$\bar{\bar{a}}_{31} + \bar{\bar{a}}_{32} = 1 \qquad (2.2)$$

$$\bar{\bar{a}}_{31}\alpha_1 + \bar{\bar{a}}_{32}\alpha_2 = 1/2 \qquad (2.3)$$

$$\bar{\bar{a}}_{31}\alpha_1^2 + \bar{\bar{a}}_{32}\alpha_2^2 = 1/3 \qquad (2.4)$$

$$\bar{a}_{31} + \bar{a}_{32} = 1/2 \qquad (2.5)$$

$$\bar{a}_{31}\alpha_1 + \bar{a}_{32}\alpha_2 = 1/6 \qquad (2.6)$$

$$a_{31} + a_{32} = 1/6 \qquad (2.7)$$

$$\bar{\bar{a}}_{31}\nu_1 + \bar{\bar{a}}_{32}\nu_2 = 1/6 \qquad (2.8)$$

Clearly the above system (2.2 – 2.8) implies that

$$\bar{a}_{3i} = \bar{\bar{a}}_{3i}(1 - \alpha_i) \text{ for } i = 1, 2. \qquad (2.9\text{–}2.10)$$

To the above system we have to supplement conditions resulting from the following:

(i) the necessary condition for the consistency of equations in (2.2–2.4), which is given by

$$1/3 - 1/2(\alpha_1 + \alpha_2) + \alpha_1\alpha_2 = 0 \qquad (2.11)$$

(ii) considering equation (3) due to Simplifying Assumption of [11] with $P = 3$ & $k = 2$, that is

$$a_{21} = \alpha_2^3/6 \qquad (2.12)$$

Along with relations (2.11–2.12), the system (2.2 – 2.8) is to be solved for the eleven parameters of the method. This system when solved in terms of three free parameters gives values for the remaining unknowns, as

$$\alpha_2 = 2 - 3\alpha_1/3(1 - 2\alpha_1), \quad \alpha_3 = 1$$

$$\bar{\bar{a}}_{32} = (2\alpha_1 - 1)/2(\alpha_1 - \alpha_2), \quad \bar{\bar{a}}_{31} = 1 - \bar{\bar{a}}_{32}$$

$$\bar{a}_{3j} = \bar{\bar{a}}_{3i}(1 - \alpha_i) \text{ for } i = 1, 2.$$

$$a_{32} = 1/6 - a_{31}, \; a_{21} = \alpha_2^3/6$$

$$\nu_2 = 1/\bar{\bar{a}}_{32}\{1/6 - \bar{\bar{a}}_{31}\nu_1\}, \quad \nu_3 = 1$$

This method is denoted by $EM_3(\alpha_1, \nu_1, a_{31})$, which is valid for $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq 1/2$.

The two cases of exception can be studied as only one special case is given below.

Special case when $\alpha_1 = \alpha_2 = 1/2$ (say) Solving the reduced system (2.2–2.3, 2.7 – 2.12) with $\alpha_1 = \alpha_2 = 1/2$ we obtain a three parameter family of solutions for the unknowns of the method (2.1), given by

$$\bar{\bar{a}}_{32} = 1 - \bar{\bar{a}}_{31}, \quad \alpha_3 = 1$$

$$\bar{a}_{3i} = \bar{\bar{a}}_{31}(1 - \alpha_i) \text{ for } i = 1, 2.$$

$$a_{32} = 1 - a_{31}, \quad a_{21} = 1/48$$

$\nu_2 = 1 / \bar{\bar{a}}_{32}(1/6 - \bar{\bar{a}}_{31}\, \nu_1)$, $\nu_3 = 1$, which is denoted as $EM_3^{(1)}(\nu_1, a_{31}, \bar{\bar{a}}_{31})$.

3. Families of fourth order Extended Nyström methods. For problem (1), a class of fourth order (explicit) EN-method be defined as

$$y_1 = y_0 + h\alpha_4 y_0' + h^2 \nu_4 y_0'' + h^3 \sum_{j=1}^{3} a_{4j} k_j + T_0(h)$$

$$y_1' = y_0' + h^2 \beta_4 y_0' + h^3 \sum_{j=1}^{3} \bar{a}_{4j} k_j + T_0'(h)$$

$$y_1'' = y_0'' + h \sum_{j=1}^{3} \bar{\bar{a}}_{4j} k_j + T_0''(h) \tag{3.1}$$

where $k_i = f(x_i + \alpha_i h,\ y_0 + h\alpha_i y_0' + h^2 \nu_i y_0'' + h^3 \sum_{j=1}^{3} a_{ij} k_j)$, $i = 1, 2, 3$.

which involves eighteen parameters, namely, $a_{31}, \bar{a}_{32}, \bar{\bar{a}}_{21}$ & $\alpha_i, \nu_i, a_{4i}, \bar{a}_{4i}, \bar{\bar{a}}_{4i}$ for $i = 1, 2, 3$.

We let each of $T_0(h)$, $T_0'(h)$ and $T_0''(h)$ to be of order $O(h^5)$. Then from table (3) of Sarma [10] there results for fourth order explicit EN-method defined by (3.1), a system of twelve equations involving fifteen parameters, given by

$$\bar{\bar{a}}_{41} + \bar{\bar{a}}_{42} + \bar{\bar{a}}_{43} = 1 \tag{3.2}$$

$$\bar{\bar{a}}_{41}\alpha_1 + \bar{\bar{a}}_{42}\alpha_2 + \bar{\bar{a}}_{43}\alpha_3 = 1/2 \tag{3.3}$$

$$\bar{\bar{a}}_{41}\alpha_1^2 + \bar{\bar{a}}_{42}\alpha_2^2 + \bar{\bar{a}}_{43}\alpha_3^2 = 1/3 \tag{3.4}$$

$$\bar{\bar{a}}_{41}\alpha_1^3 + \bar{\bar{a}}_{42}\alpha_2^3 + \bar{\bar{a}}_{43}\alpha_3^3 = 1/4 \tag{3.5}$$

$$\bar{a}_{41} + \bar{a}_{42} + \bar{a}_{43} = 1/2 \tag{3.6}$$

$$\bar{a}_{41}\alpha_1 + \bar{a}_{42}\alpha_2 + \bar{a}_{43}\alpha_3 = 1/6 \tag{3.7}$$

$$\bar{a}_{41}\alpha_1^2 + \bar{a}_{42}\alpha_2^2 + \bar{a}_{43}\alpha_3^2 = 1/12 \tag{3.8}$$

$$a_{41} + a_{42} + a_{43} = 1/6 \tag{3.9}$$

$$a_{41}\alpha_1 + a_{42}\alpha_2 + a_{43}\alpha_3 = 1/24 \tag{3.10}$$

$$\bar{\bar{a}}_{41}\nu_1 + \bar{\bar{a}}_{42}\nu_2 + \bar{\bar{a}}_{43}\nu_3 = 1/6 \tag{3.11}$$

$$\bar{\bar{a}}_{41}\alpha_1\nu_1 + \bar{\bar{a}}_{42}\alpha_2\nu_2 + \bar{\bar{a}}_{43}\alpha_3\nu_3 = 1/8 \tag{3.12}$$

$$\bar{a}_{41}\nu_1 + \bar{a}_{42}\nu_2 + \bar{a}_{43}\nu_3 = 1/24 \tag{3.13}$$

Clearly the above system (3.2 – 3.13) implies that

$$\bar{a}_{4i} = \bar{\bar{a}}_{4i}(1 - \alpha_i) \text{ for } i = 1, 2, 3. \tag{3.14–3.16}$$

As in the previous case we supplement to the above system further relations, namely,

(i) The necessary condition for the consistency of equations in (3.2 – 3.6), which is given by

$$1/4 - 1/3\,(\alpha_1 + \alpha_2 + \alpha_3) + 1/2\,(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) - \alpha_1\alpha_2\alpha_3 = 0 \tag{3.17}$$

(ii) The equation (3) (due to Simplifying Assumption) with $P = 3$ & $k = 2, 3$, turns out to be

$$a_{21} = \alpha_2^3/6 \tag{3.18}$$

$$a_{31} + a_{32} = \alpha_3^3/6 \tag{3.19}$$

Along with the relations (3.14 – 3.19), the original system (3.2 – 3.13) will be a system of thirteen equations, (3.2–3.4, 3.9–3.11, 3.13–3.19) for the eighteen parameters of the method(3.1). This system when solved in terms of free parameters a_{31} & $\alpha_1, \alpha_2, \nu_1, a_{41}$ gives values for the reamining unknowns, as

$$\alpha_3 = \frac{3 - 4(\alpha_1 + \alpha_2) + 6\alpha_1\alpha_2}{4 - 6(\alpha_1 + \alpha_2) + 12\alpha_1\alpha_2}, \quad \alpha_4 = 1$$

$$\bar{\bar{a}}_{43} = \frac{2 - 3(\alpha_1 + \alpha_2) + 6\alpha_1\alpha_2}{6(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}$$

$$\bar{\bar{a}}_{42} = \frac{2 - 3(\alpha_1 + \alpha_3) + 6\alpha_1\alpha_3}{6(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)}$$

$$\bar{\bar{a}}_{41} = 1 - \bar{\bar{a}}_{42} - \bar{\bar{a}}_{43}.$$

$$\bar{a}_{4i} = \bar{\bar{a}}_{4i}(1 - \alpha_i), \quad i = 1, 2, 3.$$

$$a_{43} = \frac{4\alpha_2(1 - 6a_{41}) + (24\,a_{41}\,\alpha_1 - 1)}{24(\alpha_2 - \alpha_3)}$$

$$a_{42} = \frac{4\alpha_3(1 - 6a_{41}) + (24\,a_{41}\,\alpha_1 - 1)}{24\,(\alpha_3 - \alpha_2)}$$

$$a_{32} = \alpha_3^3/6 - a_{31},\ a_{21} = \alpha_2^3/6$$

$$\nu_2 = \frac{4\bar{a}_{43} - \bar{\bar{a}}_{43} + 24\nu_1(\bar{a}_{41}\bar{\bar{a}}_{43} - \bar{a}_{43}\bar{\bar{a}}_{41})}{24(\bar{a}_{43}\bar{\bar{a}}_{42} - \bar{a}_{42}\bar{\bar{a}}_{42})}$$

$$\nu_3 = \frac{\bar{\bar{a}}_{42} - 4\bar{a}_{42} + 24\nu_1(\bar{a}_{41}\bar{\bar{a}}_{42} - \bar{a}_{42}\bar{\bar{a}}_{41})}{24(\bar{a}_{43}\bar{\bar{a}}_{42} - \bar{a}_{42}\bar{\bar{a}}_{43})},\quad \nu_4 = 1.$$

This method is denoted by EM_4 $(\alpha_1, \alpha_2, \nu_1, a_{31}, a_{41})$.

The above solution is valid for (i) $\alpha_1 \neq \alpha_2$ (ii) $\alpha_1 \neq \alpha_3$ (iii) $\alpha_2 \neq \alpha_3$. **The cases of exception can be studied as** only one special case where any two of α_1, α_2 & α_3 are equal.

Special case when $\alpha_1 = \alpha_2$ (say). Solving the reduced system (3.2—3.4, 3.9—3.11, 3.13—3.19) with $\alpha_1 = \alpha_2$, we obtain a five parameter family, solution for unknowns in terms of free parameters α_1, ν_1, a_{31}, a_{42}, $\bar{\bar{a}}_{42}$ are given as

$$\alpha_3 = \frac{2 - 3\alpha_1}{3(1 - 2\alpha_1)},\quad \alpha_4 = 1$$

$$\bar{a}_{43} = \frac{2\alpha_1 - 1}{2(\alpha_1 - \alpha_3)}$$

$$\bar{a}_{41} = 1 - \bar{a}_{42} - \bar{a}_{43}$$

$$\bar{\bar{a}}_{4i} = \bar{a}_{4i}(1 - \alpha_i),\quad i = 1, 2, 3.$$

$$a_{43} = \frac{4\alpha_1 - 1}{24(\alpha_1 - \alpha_3)}$$

$$a_{41} = 1/6 - a_{42} - a_{43}$$

$$a_{32} = \alpha_1^3/6 - a_{31},\ a_{21} = \alpha_1^3/6$$

$$\nu_2 = \frac{4\bar{a}_{43} - \bar{\bar{a}}_{43} + 24\nu_1(\bar{a}_{41}\bar{\bar{a}}_{43} - \bar{a}_{43}\bar{\bar{a}}_{41})}{24(\bar{a}_{43}\bar{\bar{a}}_{42} - \bar{a}_{42}\bar{\bar{a}}_{42})}$$

$$\nu_3 = \frac{\bar{\bar{a}}_{42} - 4\bar{a}_{42} + 24\nu_1(\bar{a}_{41}\bar{\bar{a}}_{42} - \bar{a}_{42}\bar{\bar{a}}_{41})}{24(\bar{a}_{43}\bar{\bar{a}}_{42} - \bar{a}_{42}\bar{\bar{a}}_{43})},\quad \nu_4 = 1.$$

This method is abbreviated as $EM_4^{(1)}(\alpha_1, \nu_1, a_{31}, a_{42}, \bar{\bar{a}}_{42})$.

4. Families of fifth order Extended Nyström methods. For the initial value problem (1), a class of fifth order (explicit) EN-method may be

defined as

$$y_1 = y_0 + h\alpha_5 y_0' + h^2\nu_5 y_0'' + h^3\sum_{j=1}^{4} a_{5j}k_j + T_0(h)$$

$$y_1' = y_0' + h^2\beta_5 y_0'' + h^3\sum_{j=1}^{4} \bar{a}_{5j}k_j + T_0'(h)$$

$$y_1'' = y_0'' + h\sum_{j=1}^{4} \bar{\bar{a}}_{5j}k_j + T_0''(h) \tag{4.1}$$

where $k_i = f(x_i + \alpha_i h,\ y_0 + h\alpha_i y_0' + h^2\nu_i y_0'' + h^3\sum_{j=1}^{i-1} a_{ij}k_j)$, $i = 1, 2, 3, 4$, **which involves twenty six parameters, namely,** a_{41}, a_{42}, a_{43}, a_{31}, a_{32}, a_{21} **and** α_i, ν_i, a_{5i}, $\bar{a}_{5i}$, $\bar{\bar{a}}_{5i}$ **for** $i = 1, 2, 3, 4$.

Letting $T_0(h)$, $T_0'(h)$ and $T_0''(h)$ each to be of order $O(h^6)$ **we obtain from table 3 of Sarma [10], for fifth order explicit** EN-method defined by (4.1), a system of nineteen equations involving twenty parameters, given by

$$\bar{\bar{a}}_{51} + \bar{\bar{a}}_{52} + \bar{\bar{a}}_{53} + \bar{\bar{a}}_{54} = 1 \tag{4.2}$$

$$\bar{\bar{a}}_{51}\alpha_1 + \bar{\bar{a}}_{52}\alpha_2 + \bar{\bar{a}}_{53}\alpha_3 + \bar{\bar{a}}_{54}\alpha_4 = 1/2 \tag{4.3}$$

$$\bar{\bar{a}}_{51}\alpha_1^2 + \bar{\bar{a}}_{52}\alpha_2^2 + \bar{\bar{a}}_{53}\alpha_3^2 + \bar{\bar{a}}_{54}\alpha_4^2 = 1/3 \tag{4.4}$$

$$\bar{\bar{a}}_{51}\alpha_1^3 + \bar{\bar{a}}_{52}\alpha_2^3 + \bar{\bar{a}}_{53}\alpha_3^3 + \bar{\bar{a}}_{54}\alpha_4^3 = 1/4 \tag{4.5}$$

$$\bar{\bar{a}}_{51}\alpha_1^4 + \bar{\bar{a}}_{52}\alpha_2^4 + \bar{\bar{a}}_{53}\alpha_3^4 + \bar{\bar{a}}_{54}\alpha_4^4 = 1/5 \tag{4.6}$$

$$\bar{a}_{51} + \bar{a}_{52} + \bar{a}_{53} + \bar{a}_{54} = 1/2 \tag{4.7}$$

$$\bar{a}_{51}\alpha_1 + \bar{a}_{52}\alpha_2 + \bar{a}_{53}\alpha_3 + \bar{a}_{54}\alpha_4 = 1/6 \tag{4.8}$$

$$\bar{a}_{51}\alpha_1^2 + \bar{a}_{52}\alpha_2^2 + \bar{a}_{53}\alpha_3^2 + \bar{a}_{54}\alpha_4^2 = 1/12 \tag{4.9}$$

$$\bar{a}_{51}\alpha_1^3 + \bar{a}_{52}\alpha_2^3 + \bar{a}_{53}\alpha_3^3 + \bar{a}_{54}\alpha_4^3 = 1/20 \tag{4.10}$$

$$a_{51} + a_{52} + a_{53} + a_{54} = 1/6 \tag{4.11}$$

$$a_{51}\alpha_1 + a_{52}\alpha_2 + a_{53}\alpha_3 + a_{54}\alpha_4 = 1/24 \tag{4.12}$$

$$a_{51}\alpha_1^2 + a_{52}\alpha_2^2 + a_{53}\alpha_3^2 + a_{54}\alpha_4^2 = 1/60 \tag{4.13}$$

$$\bar{\bar{a}}_{51}\nu_1 + \bar{\bar{a}}_{52}\nu_2 + \bar{\bar{a}}_{53}\nu_3 + \bar{\bar{a}}_{54}\nu_4 = 1/6 \tag{4.14}$$

$$\bar{\bar{a}}_{51}\alpha_1\nu_1 + \bar{\bar{a}}_{52}\alpha_2\nu_2 + \bar{\bar{a}}_{53}\alpha_3\nu_3 + \bar{\bar{a}}_{54}\alpha_4\nu_4 = 1/8 \tag{4.15}$$

$$\bar{\bar{a}}_{51}\, \nu_1^2 + \bar{\bar{a}}_{52}\, \nu_2^2 + \bar{\bar{a}}_{53}\, \nu_3^2 + \bar{\bar{a}}_{54}\, \alpha_4^2 = 1/20 \tag{4.16}$$

$$\bar{\bar{a}}_{51}\, \alpha_1^2\nu_1 + \bar{\bar{a}}_{52}\alpha_2^2\nu_2 + \bar{\bar{a}}_{53}\alpha_3^2\nu_3 + \bar{\bar{a}}_{54}\, \alpha_4^2\nu_4 = 1/10 \tag{4.17}$$

$$\bar{a}_{51}\, \nu_1 + \bar{a}_{52}\, \nu_2 + \bar{a}_{53}\, \nu_3 + \bar{a}_{54}\, \nu_4 = 1/24 \tag{4.18}$$

$$\bar{a}_{51}\, \alpha_1\nu_1 + \bar{a}_{52}\alpha_2\nu_2 + \bar{a}_{53}\alpha_3\nu_2 + \bar{a}_{54}\, \alpha_4\nu_2 = 1/40 \tag{4.19}$$

$$a_{51}\nu_1 + a_{52}\nu_2 + a_{53}\nu_3 + a_{54}\, \nu_4 = 1/120 \tag{4.20}$$

Obviously the above system (4.2 — 4.20) implies that

$$\bar{a}_{5i} + \bar{\bar{a}}_{5i}\,(1 - \alpha_i) \text{ for } i = 1,\ 2,\ 3,\ 4. \tag{4.21— 4.24}$$

As in the previous, here too we supplement to the above system further relations namely, (i) the necessary condition for the consistency of equations in (4.2 — 4.6), which is given by

$$1/5 - 1/4\,(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + 1/3\,(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\,\alpha_4 + \alpha_2\alpha_4$$

$$+ \alpha_3\alpha_4) - 1/2(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4) + \alpha_1\alpha_2\alpha_3\alpha_4 = 0 \tag{4.25}$$

(ii) the equation (3) (due to Simplifying Assumption) with

$P = 3$ & $k = 2,\ 3,\ 4$ turning out to be,

$$a_{21} = \alpha_2^3/6 \tag{4.26}$$

$$a_{31} + a_{32} = \alpha_3^3/6 \tag{4.27}$$

$$a_{41} + a_{42} + a_{43} = \alpha_4^3/6 \tag{4.28}$$

Along with the relations (4.21 — 4.28), the original system (4.2 — 4.20) will be a system of eighteen equations, namely (4.2—4.5, 4.11—4.14, 4.18, 4.20 — 4.28) for the twenty six parameters of the method (4.1). This reduced system when solved in terms of eight free parameters, namely, α_1, α_2, α_3, ν_1, a_{31}, a_{41}, a_{42}, a_{51}, gives the values of the remaining unknowns as

$$\alpha_4 = \frac{12 - 15(\alpha_1 + \alpha_2 + \alpha_3) + 20(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) - 30(\alpha_1\alpha_2\alpha_3)}{15 - 20(\alpha_1 + \alpha_2 + \alpha_3) + 30(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) - 60(\alpha_1\alpha_2\alpha_3)}$$

$$\alpha_5 = 1$$

$$a_{52} = \frac{20\alpha_3\alpha_4(1 - 6a_{51}) + 5(\alpha_3 + \alpha_4)\,(24a_{51}\alpha_1 - 1) + 2(1 - 60a_{51}\alpha_1^2)}{120(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)}$$

$$a_{53} = \frac{20\alpha_2\alpha_4(1 - 6a_{51}) + 5(\alpha_2 + \alpha_4)(24a_{51}\alpha_1 - 1) + 2(1 - 60a_{51}\alpha_1^2)}{120(\alpha_2 - \alpha_3)\,(\alpha_4 - \alpha_3)}$$

$$a_{54} = \frac{20\alpha_3\alpha_2(1 - 6a_{51}) + 5(\alpha_3 + \alpha_2)(24a_{51}\alpha_1 - 1) + 2(1 - 60a_{51}\alpha_1^2)}{120(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_4)}$$

$$\bar{\bar{a}}_{52} = \frac{4(\alpha_1 + \alpha_4 + \alpha_3) - 6(\alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_4\alpha_3) + 12\alpha_1\alpha_4\alpha_3 - 3}{12(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)}$$

$$\bar{\bar{a}}_{53} = \frac{4(\alpha_1 + \alpha_4 + \alpha_2) - 6(\alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_4\alpha_2) + 12\alpha_1\alpha_4\alpha_2 - 3}{12(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(\alpha_4 - \alpha_3)}$$

$$\bar{\bar{a}}_{54} = \frac{4(\alpha_1 + \alpha_2 + \alpha_3) - 6(\alpha_1\alpha_3 + \alpha_1\alpha_2 + \alpha_2\alpha_3) + 12\alpha_1\alpha_2\alpha_3 - 3}{12(\alpha_1 - \alpha_4)\,(\alpha_3 - \alpha_4)(\alpha_2 - \alpha_4)}$$

$$\bar{\bar{a}}_{51} = 1 - \bar{\bar{a}}_{52} - \bar{\bar{a}}_{53} - \bar{\bar{a}}_{54}$$

$$\bar{a}_{5i} = \bar{\bar{a}}_{5i}(1 - \alpha_i), \qquad i = 1,\ 2,\ 3.\ 4.$$

$$a_{43} = \alpha_4^3/6 - a_{41} - a_{42}$$

$$a_{32} = \alpha_3^3/6 - a_{31}, \qquad a_{21} = \alpha_2^3/6$$

$$A = \bar{\bar{a}}_{53}\bar{\bar{a}}_{54}(5a_{53} - \bar{a}_{53} - 120\nu_1\,(\bar{a}_{53}a_{51} - \bar{a}_{51}a_{53}))$$

$$B = 20\bar{\bar{a}}_{53}(a_{54}\bar{a}_{53} - a_{53}\bar{a}_{54})(1 + 6\nu_1\bar{\bar{a}}_{51}) - \bar{\bar{a}}_{53}^2(5a_{54} - \bar{a}_{54} +$$
$$+ 120\nu_1(\bar{a}_{54}a_{51} - a_{54}\bar{a}_{51})$$

$$C = 120\{\bar{\bar{a}}_{54}\bar{\bar{a}}_{53}(a_{53}\bar{a}_{52} - \bar{a}_{53}a_{52}) + \bar{\bar{a}}_{52}\bar{\bar{a}}_{53}(a_{54}\bar{a}_{53} - a_{53}\bar{a}_{54}) -$$
$$- \bar{\bar{a}}_{53}^2(a_{54}a_{52} - a_{52}a_{54})$$

$$A1 = \bar{\bar{a}}_{52}\bar{\bar{a}}_{54}(5a_{52} - \bar{a}_{53} - 120\nu_1(\bar{a}_{52}a_{51} - \bar{a}_{51}a_{52}))$$

$$B1 = 20\bar{\bar{a}}_{52}(a_{54}\bar{a}_{52} - a_{52}\bar{a}_{54})(1 + 6\nu_1\bar{\bar{a}}_{51}) - \bar{\bar{a}}_{52}^2(5a_{54} - \bar{a}_{54} +$$
$$120\nu_1(\bar{a}_{54}a_{51} - a_{54}\bar{a}_{51}))$$

$$C1 = 120\{\bar{\bar{a}}_{54}\bar{\bar{a}}_{52}(\bar{a}_{53}a_{52} - a_{53}\bar{a}_{52}) + \bar{\bar{a}}_{52}\bar{\bar{a}}_{53}(a_{54}\bar{a}_{52} - a_{52}\bar{a}_{54}) - \bar{\bar{a}}_{52}^2)$$
$$(a_{54}\bar{a}_{53} - a_{53}\bar{a}_{54})\}$$

$$A2 = \bar{\bar{a}}_{53}\bar{\bar{a}}_{52}\,(5a_{52} - \bar{a}_{52} - 120\nu_1(\bar{a}_{52}a_{51} - \bar{a}_{51}a_{52}))$$

$$B2 = 20\bar{\bar{a}}_{52}(a_{53}\bar{a}_{52} - \bar{a}_{53}a_{52})(1 + 6\nu_1 \bar{\bar{a}}_{51}) - \bar{\bar{a}}_{52}^2(5a_{54} - \bar{a}_{54} +$$

$$+120\nu_1(\bar{a}_{54}a_{51} - a_{54}\bar{a}_{51}))$$

$$C2 = 120\{\bar{\bar{a}}_{52}\bar{\bar{a}}_{53}(a_{54}\bar{a}_{52} - \bar{a}_{54}a_{52}) + \bar{\bar{a}}_{52}\bar{\bar{a}}_{54}\ (a_{52}\bar{a}_{53} - a_{53}\bar{a}_{52}) -$$

$$\bar{\bar{a}}_{52}^2(a_{54}a_{53} - a_{53}a_{54})\}$$

$$\nu_2 = (A + B)/C$$

$$\nu_3 = (A1 + B1)/C1$$

$$\nu_4 = (A2 + B2)\ /\ C2, \qquad \nu_5 = 1.$$

This method is denoted by $EM_5\ (\alpha_1, \alpha_2, \alpha_3, \nu_1, a_{31}, a_{41}, a_{42}, a_{51})$.

The above solution is valid for (i) $\alpha_1 \neq \alpha_2$ (ii) $\alpha_1 \neq \alpha_3$ (iii) $\alpha_2 \neq \alpha_3$ (iv) $\alpha_1 \neq \alpha_4$ (v) $\alpha_2 \neq \alpha_4$ (vi) $\alpha_3 \neq \alpha_4$. The cases of exception can be studied as three special cases, as given below

1. Any two of α_1, α_2, α_3 & α_4 are equal.
2. Any three of α_1, α_2, α_3 & α_4 are equal.
3. Any pairwise distinct combinations of α_1, α_2, α_3 & α_4 are equal.

(a) Special case when $\alpha_1 = \alpha_2$ (say). Solving the reduced system with $\alpha_1 = \alpha_2$, we obtain a eight parameter family, the unknowns in terms of free parameters $\alpha_1, \alpha_3, \nu_1, a_{31}, a_{41}, a_{42}, a_{52}, \bar{\bar{a}}_{52}$ are given by

$$\alpha_4 = \frac{3 - 4(\alpha_1 + \alpha_3) + 6(\alpha_1\alpha_3)}{4 - 6(\alpha_1 + \alpha_3) + 12(\alpha_1\alpha_3)} \qquad \alpha_5 = 1$$

$$\bar{\bar{a}}_{54} = \frac{2 - 3(\alpha_1 + \alpha_3) + 6(\alpha_1\alpha_3)}{6(\alpha_1 - \alpha_4)(\alpha_3 - \alpha_4)}$$

$$\bar{\bar{a}}_{53} = \frac{2 - 3(\alpha_1 + \alpha_4) + 6(\alpha_1\alpha_4)}{6(\alpha_1 - \alpha_3)(\alpha_4 - \alpha_3)}$$

$$\bar{\bar{a}}_{51} = 1 - \bar{\bar{a}}_{52} - \bar{\bar{a}}_{53} - \bar{\bar{a}}_{54}$$

$$\bar{a}_{5i} = \bar{\bar{a}}_{5i}(1 - \alpha_i), \qquad i = 1, 2, 3, 4.$$

$$a_{53} = \frac{2 - 5(\alpha_1 + \alpha_4) + 20(\alpha_1\alpha_4)}{120(\alpha_3 - \alpha_4)}$$

$$a_{54} = \frac{2 - 5(\alpha_1 + \alpha_3) + 20(\alpha_1\alpha_3)}{120(\alpha_3 - \alpha_4)}$$

$$a_{51} = 1/6 - a_{52} - a_{53} - a_{54}$$

$$a_{43} = \alpha_4^3/6 - a_{41} - a_{42}$$

$$a_{32} = \alpha_3^3/6 - a_{31}, \ a_{21} = \alpha_1^3/6$$

$$\nu_2 = (A + B)/C$$

$$\nu_3 = (A1 + B1)/C1$$

$$\nu_4 = (A2 + B2)\ /\ C2, \qquad \nu_5 = 1$$

This method is denoted by $EM_5^{(1)}\ (\alpha_1, \alpha_3, \nu_1, a_{31}, a_{41}, a_{42}, \bar{\bar{a}}_{52})$ valid for $\alpha_1 \neq \alpha_2 \neq \alpha_3 = \alpha_4$.

(b) Special case when three out of four α_i are equal. Solving the reduced system with $\alpha_1 = \alpha_2 = \alpha_3$ (say), we obtain a nine parameter family, the values of the unknowns in terms of free parameters α_1, ν_1, $a_{31}, a_{41}, a_{42}, a_{52}, a_{53}, \bar{\bar{a}}_{52}, \bar{\bar{a}}_{53}$ are given by

$$\alpha_4 = \frac{2 - 3\alpha_1}{3(1 - 2\alpha_1)}$$

$$\bar{\bar{a}}_{54} = \frac{2\alpha_1 - 1}{2(\alpha_1 - \alpha_4)}$$

$$\bar{\bar{a}}_{51} = 1 - \bar{\bar{a}}_{52} - \bar{\bar{a}}_{53} - \bar{\bar{a}}_{54}$$

$$\bar{a}_{5i} = \bar{\bar{a}}_{5i}(1 - \alpha_i), \qquad i = 1, 2, 3, 4.$$

$$a_{54} = \frac{4\alpha_1 - 1}{24(\alpha_1 - \alpha_4)}$$

$$a_{51} = 1/6 - a_{52} - a_{53} - a_{54}$$

$$a_{43} = \alpha_4^3/6 - a_{41} - a_{42}$$

$$a_{32} = \alpha_1^3/6 - a_{31}, \quad a_{21} = \alpha_1^3/6$$

$$\nu_2 = (A + B)/C$$

$$\nu_3 = (A1 + B1)/C1$$

$$\nu_4 = (A2 + B2)/C2, \quad \nu_5 = 1$$

This method is denoted by $EM_5^{(2)}(\alpha_1, \nu_1, a_{31}, a_{41}, a_{42}, a_{52}, a_{53}, \bar{\bar{a}}_{52}, \bar{\bar{a}}_{53})$.

(c) Special case when $\alpha_1 = \alpha_2$ & $\alpha_3 = \alpha_4$ (say). Solving the system (4.2 – 4.5, 4.11–4.14, 4.18, 4.20–4.28) with $\alpha_1 = \alpha_2$ & $\alpha_3 = \alpha_4$ we obtain a nine parameter family, the values of the remaining unknowns in terms of free parameters $\alpha_1, \nu_1, a_{31}, a_{41}, a_{42}, a_{52}, a_{53}, \bar{\bar{a}}_{52}, \bar{\bar{a}}_{53}$ are given by

$$\alpha_4 = \frac{2 - 3\alpha_1}{3(1 - 2\alpha_1)}, \quad \alpha_5 = 1$$

$$\bar{\bar{a}}_{54} = \frac{2\alpha_1(1 - \bar{\bar{a}}_{53}) + 2\bar{\bar{a}}_{53}\alpha_4 - 1}{2(\alpha_1 - \alpha_3)}$$

$$\bar{\bar{a}}_{51} = 1 - \bar{\bar{a}}_{52} - \bar{\bar{a}}_{53} - \bar{\bar{a}}_{54}$$

$$\bar{a}_{5i} = \bar{\bar{a}}_{5i}(1 - \alpha_i), \quad i = 1, 2, 3, 4.$$

$$a_{54} = \frac{4\alpha_1(1 - 6a_{53}) + 24a_{53}\alpha_4 - 1}{24(\alpha_1 - \alpha_4)}$$

$$a_{51} = 1/6 - a_{52} - a_{53} - a_{54}$$

$$a_{43} = \alpha_4^3/6 - a_{41} - a_{42}$$

$$a_{32} = \alpha_1^3/6 - a_{31}, \; a_{21} = \alpha_1^3/6$$

$$\nu_2 = (A + B)/C$$

$$\nu_3 = (A1 + B1)/C1$$

$$\nu_4 = (A2 + B2)/C2, \quad \nu_5 = 1$$

This method is denoted by $EM_5^{(3)}(\alpha_1, \nu_1, a_{31}, a_{41}, a_{42}, a_{52}, a_{53}, \bar{\bar{a}}_{52}, \bar{\bar{a}}_{53})$.

Applications of these methods to solving special cases of problems like (1) is set aside for report in another paper.

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