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SOME REMARKS ON MEANS

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1. Introduction. A mean (of two positive real numbers) is defined in [18] as a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ which has the property:

(1) $$\min (x, y) \leqslant M(x, y) \leqslant \max (x, y), \ \forall x, y \geqslant 0.$$

Most of the usual means have additional properties like symmetry:

(2) $$M(x, y) = M(y, x), \ \forall x, y > 0;$$

homogeneity (of order one):

(3) $$M(tx, ty) = tM(x, y), \ \forall t, x, y > 0;$$

or monotony: the mean M is increasing if:

(4) $$M(x, y) \leqslant M(x', y') \text{ if } x \leqslant x' \text{ and } y \leqslant y'.$$

We must remark that the relation (1) does not imply any kind of monotony. For example, the counterharmonic mean C, defined by:

$$C(x, y) = (x^2 + y^2) / (x + y)$$

has the property:

$$C(1/3,1) = C(1/2, 1) = 5/6.$$

Some means have appeared in practical problems. Naturally, the most employed is the arithmetic mean but, as it is pointed out in [10], in the theory of electrical potential it is used the harmonic mean while in investment problems the geometric mean. Also, in [30], it is mentioned the role of the logarithmic mean in the study of the distribution of electrical charge on a conductor.

Apart from these means, there are also defined many others. In fact we dispose of methods of contruction of means. In what follows we shall remind a few of them and also some of the generated means.

Then we summarize the main relations between means and apply them to the study of a property of homogeneity more general than (3). Finally we give results on double sequences related to the arithmetic-geometric mean of Gauss.

The paper contains some new results but its main aim is to point out the results of the author which have appeared in preprints or in volumes with less circulation.

2. Construction of means. There are more general methods for construction of means. Some of them use algebraic methods, or functional equations like those from [6] [24], [30]. Others are based on the mean value theorem for derivatives or for integrals (see [12], [30], [22], [9] for more references). In [35] we have used the mean value theorem for integrals (with two functions) which seems to be more convenient than the mean value theorem of Cauchy used in [9]. If f is a monotone and continuous function which does not change the sign and is not identical zero on any interval, then we define a mean $V_{f,g}$ by:

$$V_{f,g}(x, y) = f^{-1}\left(\int_x^y (f(t)\, g(t)\, dt \,/ \int_x^y g(t)\, dt\right).$$

We can obtain on this way some of the usual means like:

$$S_{r,s}(x, y) = ((s/r)\,(x^r - y^r)\,/\,(x^s - y^s))^{1/(r-s)} \quad \text{(extended means)}$$

and

$$L_r(x, y) = S_{r,0}(x, y) = (r^{-1}(x^r - y^r)\,/\,(\log x - \log y))^{1/r}$$
(generalized logarithmic).

To get them we use: $f(t) = t^{r-s}$ and $g(t) = t^{s-1}$, with $s = 0$ in the second case. As special cases we have:

$$S_{2r,r}(x, y) = P_r(x, y) = ((x^r + y^r)/2)^{1/r} \quad \text{(the power mean)}$$

$$A(x, y) = P_1(x, y) = (x + y)/2 \quad \text{(the arithmetic mean)}$$

$$G(x, y) = P_0(x, y) = (xy)^{1/2} \quad \text{(the geometric mean)}$$

$$H(x, y) = P_{-1}(x, y) = 2xy/(x + y) \quad \text{(the harmonic mean)}$$

$$L(x, y) = L_1(x, y) = (x - y)/(\log x - \log y) \quad \text{(the logarithmic mean)}$$

$$S_r(x, y) = S_{r,1}(x, y) = (r^{-1}(x^r - y^r)/(x - y))^{1/(r-1)} \quad \text{(the Galvani mean)}$$

$$I(x, y) = S_1(x, y) = e^{-1}(x^x/y^y)^{1/(x-y)} \quad \text{(the identric mean)}$$

$$I_r(x, y) = S_{r,r}(x, y) = I^{1/r}(x^r, y^r) \quad \text{(generalized identric mean).}$$

Also, taking $f(t) = t$, $g(t) = e^t$, we get an exponential mean:

$$E(x, y) = ((x - 1)e^x - (y - 1)e^y/(e^x - e^y)$$

which we defined in [36] and studied also in [28].

Remarking that almost all of the above means are of the form:

$$W_{r,s}(x, y) = \left(\frac{g_s(1,1)}{f_r(1,1)}\,\frac{f_r(x,y)}{g_s(x,y)}\right)^{1/(r-s)}$$

where f_r and g_s are homogeneous functions of degree r respectively s subject to some conditions, we can give also the following examples:

$$M_r(x, y) = (xy^r + yx^r)\,/\,(x^r + y^r) \quad \text{(the Moskovitz mean)}$$

$$P_{r,s}(x, y) = ((x^r + y^r)\,/\,(x^s + y^s))^{1/(r-s)} \quad \text{(the Gini mean)}$$

with special cases:

$P_{r,r-1}$ (the Beckenbach – Lehmer mean)

$C = P_{2,1}$ (the contraharmonic mean).

For $r = s$, the Gini mean is given in [2] by:

$$P_{r,r}(x, y) = \exp\,((x^r \log x + y^r \log y)\,/\,(x^r + y^r)).$$

Using the same idea, we have defined in [37] some generalizations of geometric and harmonic means. There are well known the nonsymmetric generalizations of A, G and H;

$$A_t(x, y) = tx + (1 - t)y, \quad G_t(x, y) = x^t y^{1-t},$$

$$H_t(x, y) = xy\,/\,(ty + (1 - t)x) \text{ for } t \in [0,1].$$

We have proposed the following (symmetric or not) generalizations o G and H:

$$G_{a,b}(x, y) = (ax^2 + (1 - a - b)xy + by^2)^{1/2}, \quad 0 \leq a,\ b \leq 1$$

$$H_{a,b,c,d}(x, y) = (ax^2 + (c + d - a - b)xy + by^2)/(cx + dy), \; 0 \leq a \leq c,\ 0 \leq b \leq d$$

We remind also that Tricomi has characterized in [41] the linear combinations $aA + bG + (1 - a - b)H$ which are means.

Finally we must remark that from some given means we can construct others by various methods. For example, to a nonsymmetric mean M we can attach a symmetric one M^s defined by:

$$M^s(x, y) = (M(x, y) + M(y, x))/2.$$

Also, for a given mean M we can consider its inverse M^i defined by:

$$M^i(x, y) = xy/M(x, y)$$

or its complement M^c defined by:

$$M^c(x, y) = x + y - M(x, y).$$

For instance, the inverse of G is G, that of A is H and of L is L_{-1}. The complement of A is A and of H is C.

Given the mean M and the bijection f, we can define a mean $M(f)$ by:

$$M(f)\,(x, y) = f^{-1}(M(f(x)), f(y)).$$

This method was studied for example by Pietra in [26]. It is easy to see that from the arithmetic mean we can obtain: the harmonic mean H using $f(x) = 1/x$, the geometric mean G using $f(x) = \log x$, the power mean P_r, using $f(x) = x^r$ and so on. Also, in [28] we have remarked that:

$$E = I(\exp). \tag{5}$$

In what follows we shall also use two means obtained this way:

$$F = L(\exp) \text{ and } R_r = P_r(\exp).$$

We denote also: $R = R_1$.

Given two means M and M' we can also define the means $M \vee M'$ and $M \wedge M'$ by:

$$M \vee M'(x, y) = \max\,(M(x, y), M'(x, y))$$

respectively:

$$M \wedge M'(x, y) = \min\,(M(x, y), M'(x, y)).$$

Their meaning will be more clear in the following paragraph.

Also, given three means M, M', M'' we define the mean $M(M', M'')$ by:

$$M(M', M'')\,(x, y) = M(M'(x, y), M''(x, y)).$$

The means can be even the first or the second projections given by:

$$P'(x, y) = x,\ P''(x, y) = y.$$

This method was used, for instance, in [11], [16] or [17]. Means obtained by this method are also used in [12], [3] and [4]. In [8] it is studied the problem of conservation by composition of a given class of means. For example, when $P_r(P_s, P_t)$ is again a power mean?

We say that M'' is symmetric to M' relative to M if:

$$M(M', M'') = M.$$

The symmetry relative to A is considered in [23] related to some results of Tricomi [40]. In fact, M' is symmetric to M'' relative to A if M'' is the complement of A and M' is symmetric to M'' relative to G if M'' is the inverse of M'.

Another method of construction of means will be given in the last paragraph.

3. Comparison of means. Existing more means it was natural to compare them. The beginning was done by the famous geometric-arithmetic mean inequality which has many proofs (see [9]).

We say that the means M and M' are in the relation: $M < M'$ if

$$M(x, y) < M'(x, y) \text{ for } x \neq y.$$

As it is known (see [18]):

$$P_r < P_s \text{ if } r < s,$$

which generalizes the above mentioned inequality:

$$H < G < A.$$

We also say that P_r is strictly increasing in r. Analogously, in [30] K. B. Stolarsky proved that S_r is strictly increasing in r and that $S_{r,s}$ is strictly increasing in both r and s. This last result was improved by E. B. Leach and M. C. Sholander in [20] comparing $S_{r,s}$ with $S_{u,v}$.

Passing to the comparison between means from different classes, we begin with the result of A. O. Pittenger from [27]: for any $r \neq 0$ we have

$$P_s < S_r < P_t$$

where, if we denote: $r_1 = (r+1)/3$ and $r_2 = (r-1)\log 2 / \log r$ for $r > 0$, $r \neq 1$ and $r_2 = \log 2$ if $r = 1$, we have $s = \min\,(r_1, r_2)$, $t = \max(r_1, r_2)$ if $r > 0$ and $s = \min\,(0, r_1)$, $t = \max\,(0, r_1)$ if $r < 0$. Of course, we get the equality for $r = 2$, $1/2$ or -1 because:

$$S_2 = A = P_1,\ S_{1/2} = P_{1/2},\ S_{-1} = G = P_0.$$

For $r = 0$ we have:

$$P_0 = G < L < P_{1/3} \tag{6}$$

which was proved by Lin in [21]. For $r = 1$ there also follows:

$$P_{2/3} < I < P_{\log 2}. \tag{7}$$

In all the cases the values of s and of t are sharp, that is S_r is not comparable with P_u for $s < u < t$.

Putting in (6) and (7) $x = u^r$ and $y = v^r$ we get:

$$G < L_r < P_{r/3} < P_{2r/3} < I_r < P_{r\log 2},\ r > 0.$$

In [36] we have proved that:

$$E > A = P_1$$

and that E is not comparable with P_r for $r > 5/3$. We have conjectured that $E > P_{5/3}$.

In [28] using the remark (5) and putting in (7) $x = e^u$, $y = e^v$ and logarithmating, we get:

$$R_{2/3} < E < R_{\log 2}.$$

Also, from (6) and (7), we have $G < L < I$ which gives, as above:

$$A < F < E. \tag{8}$$

Relations between generalized arithmetic, geometric and harmonic means have been given in [37]. We remind some of them:

$$G_{a,b} < G_{a',b'} \Leftrightarrow a' - a = b' - b > 0$$

$$H_{a,b,c,d} < H_{a',b',c',d'} \Leftrightarrow a/c < a'/c',\ b/d < b'/d',\ (c - a + b)/(c + d) = = (c' - a' + b')/(c' + d')$$

$$G_{a,b} < A_c \Leftrightarrow \sqrt{a} + \sqrt{b} < 1,\ c = (1 + a - b)/2$$

$$H_{a,b,c,d} < A_e \Leftrightarrow a/c + b/d < 1,\ e = (a - b + d)/(c + d).$$

There are also inequalities of mixed type, which involve more means. So, in [12] B. C. Carlson proved that:

$$G \cdot P_{1/2} < L^2.$$

This was improved by H. Alzer in [4] as:

$$G \cdot I < L^2$$

who also proved in [3] that:

$$A \cdot G < L \cdot I.$$

Using them, we have derived in [27] the inequalities:

$$A + R - F < E < 2F - A$$

which can be compared with (8).

For nonsymmetric means, in [29] and [16] was used a comparability on a subset. The means M and M' are in relation $M \prec_D M'$ if:

$$M(x, y) \leqslant M'(x, y) \text{ for } (x, y) \in D$$

where usually $D = D' = \{(x, y);\ 0 < x < y\}$ or $D = D'' = \{(x, y);\ 0 < y < x\}$. In what follows we use the weak comparability we have considered in [32]: $M \prec M'$ if $M \prec_{D'} M'$ and $M' \prec_{D''} M$. For example, if $p < q$ we have $A_q \prec A_p$, $G_q \prec G_p$ and $H_q \prec H_p$. Also, in [37] we have proved the following relations:

$$G_{a,b} \prec G_{a',b'} \Leftrightarrow a \geqslant a',\ b \leqslant b'$$

$$G_{a,b} \prec A_c \Leftrightarrow c \leqslant \min (\sqrt{a},\ 1 - \sqrt{b})$$

$$H_{a,b,c,d} \prec A_e \Leftrightarrow e \leqslant \min (a/c,\ (a + d - b)/(c + d),\ 1 - b/d).$$

They are immediate the following equivalences:

$$M \prec M' \Leftrightarrow M'_l \prec M_l \Leftrightarrow M'_c \prec M_c \Leftrightarrow M \vee M' = M' \Leftrightarrow$$
$$\Leftrightarrow M \wedge M' = M \Leftrightarrow M \prec M'' \ (M, M') \prec M',\ \forall M''.$$

We also have:

$$M(P', M') \prec M' \text{ and } M(M', P'') \succ M'.$$

Finally, we remark that if we denote by:

$$D(M) = \{(x, y);\ M(x, y) = 1\}$$

and there is an x such that $(x, y) \in D(M)$ while $(x, y') \in D(M')$, then we have:

$$y > y' \text{ if } M < M'$$

but

$$y > y' \text{ for } x > 1 \text{ and } y < y' \text{ for } x < 1 \text{ if } M \prec M'.$$

4. Homogeneity properties. Remarking that not all the studied means are homogeneous or logarithmic — homogeneous, that is:

$$M(x^t, y^t) = M'(x, y) \text{ for } x, y, t > 0$$

in [28] we have defined the following generalization: for a given $t > 0$ the mean M is called t-subhomogeneous (t-log-subhomogeneous) if:

$$M(tx, ty) \leqslant tM(x, y) \text{ (respectively } M(x^t, y^t) \leqslant M^t(x, y)).$$

If the inequalities are reversed, the mean is called t-superhomogeneous, respectively t-log-superhomogeneous. Of course, if M is t-subhomogeneous it is $1/t$-superhomogeneous.

We have the following results.

LEMMA. a) *If M is t-log-subhomogeneous, then M^t is t-log-superhomogeneous.*

b) *If M is t-subhomogeneous, then M^c is t-superhomogeneous.*

c) *If M is t-subhomogeneous, then M^s is also t-subhomogeneous.*

d) *If M, M', M'' are t-(log)-subhomogeneous and M is increasing, then $M(M', M'')$ is t-(log)-subhomogeneous.*

We remark that the lemma imposes another way of obtaining a symmetric mean from a given one which preserves the t-log-subhomogeneity. This can be defined by:

$$M^g(x, y) = G(M(x, y), M(y, x)).$$

In [28] we have remarked that the means P_r are t-log-subhomogeneous for $t < 1$ and R_r are t-subhomogeneous for $t < 1$. Also we have proved that E is t-subhomogeneous and I is t-log-subhomogeneous for $t \leq 2 / \log 8 = 0.961\ldots$ Using the same method we can prove the following more general result:

THEOREM 1. *If the mean M verifies the double relation:*

9) $$P_p < M < P_q$$

hen it is p/q — log-subhomogeneous.

Proof. The inequality (9) means that:

$$((x^p + y^p)/2)^{1/p} < M(x, y) < ((x^q + y^q)/2)^{1/q}.$$

Putting in the first part $x^p = u$ and $y^p = v$, we get:

$$(u + v)/2 < M^p(u^{1/p}, v^{1/p}).$$

Also, from the second part, for $x^q = u$, $y^q = v$, we have:

$$M^q(u^{1/q}, v^{1/q}) < (u + v)/2.$$

Taking, in the resulting inequality, $a = u^{1/p}$ and $b = v^{1/p}$ it follows:

$$M(a^{p/q}, b^{p/q}) < M^{p/q}(a, b).$$

Analogously, we can prove:

THEOREM 2. *If*:

$$R_p < M < R_q$$

then M is p/q-subhomogeneous.

5. Double sequences. Given two means M and M' one can define a double sequence (or a bidimensional iteration algorithm) starting from two positive numbers x_0 and y_0, in two ways:

(10) $$x_{n+1} = M(x_n, y_n), \; y_{n+1} = M'(x_n, y_n)$$

or

(11) $$x_{n+1} = M(x_n, y_n), \; y_{n+1} = M'(x_{n+1}, y_n).$$

The first algorithm is known under the name of Gauss thought it was defined by Lagrange in [19] (for $M = A$ and $M' = G$). The second algorithm takes in [14] the name of Archimedes because his procedure for estimating π can be interpreted by (11) with $M = H$ and $M' = G$. But it is also known as Schwab or Borchard or Pfaff's algorithm (at least for $M = A$ and $M' = G$). As it is pointed out in [17] the algorithm (11) can be deduced from (10) taking $M'(M, P'')$ for M'.

The basic problem for these algorithms is related to the circumstances in which the sequences (x_n) and (y_n) converge to a common limit. As follows from the simple example given in [41] by $M = P''$ and $M' = P'$, that is:

$$x_{n+1} = y_n, \quad y_{n+1} = x_n$$

this convergence does not occur without supplementary conditions. If the sequences (x_n) and (y_n) have a common limit for any starting points x_0, y_0, we denote the common limit by $M\boxed{g}M'(x_0, y_0)$ if we apply the algorithm (10) and by $M\boxed{a}M'(x_0, y_0)$ if it is used (11). In both cases it results a mean (compound mean). If $M\boxed{g}M'$ (or $M\boxed{a}M'$) exists we shall say that M and M' are G-composible (respectively A-composible).

In [14] it was proved that two continuous symmetric means are A-composible if they have the property:

(12) $$M(x, y) = x \Rightarrow x = y.$$

In [32] we have proved the same result renouncing at the symmetry of means. In [31] we consider also the complex case and in [34] the case of lattices.

The study was more complicated in the case of G-composition. This was begun with the case of homogeneous means in [5], [25], [1] and [41]. Then it was proved for comparable or weakly comparable means which are continuous and satisfy (12), in [29], [16], [42] and [32]. In [38] we renounced at comparability for symmetric means. This was done by using the means $M \wedge M'$ and $M \vee M'$ and the double sequence generated by them. The proof is so very simple. With a more sophisticated method the result was proved for nonsymmetric means in [17]. Thus, if M and M' are continuous means with the property (12) then $M\boxed{g}M'$ exists. In [39] we have also renounced at these properties for one of the two means. We sketch here the proof of this result.

THEOREM 3. *If one of the means M and M' is continuous and satisfies* (12) *and*

(13) $$M(x, y) = y \Rightarrow x = y.$$

then $M\boxed{g}M'$ exists.

Proof. From (1) follows that the sequences (x_n) and (y_n) given by (10) lie in the closed interval I determined by x_0 and y_0. By the Bolzano-Weierstrass theorem, there is a sequence (n_k) and the points x, y, x', y', such that:

$$\lim_{k\to\infty} x_{n_k} = x,\ \lim_{k\to\infty} y_{n_k} = y,\ \lim_{k\to\infty} x_{n_k+1} = x',\ \lim_{k\to\infty} y_{n_k+1} = y'$$

Then $x = y$. Indeed, if we suppose $x < y$, as $x_{n_k+1} = M(x_{n_k}, y_{n_k})$ and $y_{n_k+1} = M'(x_{n_k}, y_{n_k})$, it follows that:

$$x \leqslant x' \leqslant y,\ x \leqslant y' \leqslant y.$$

We prove that:

(14) $$x' = x \text{ or } x' = y.$$

If $x < x' \leqslant y' \leqslant y$, we choose $0 < r < (x' - x)/2$ and $K = K_r$ such that:

$$|x_{n_k+1} - x'| < r \text{ and } |y_{n_k+1} - y'| < r.$$

So:

$$x_{n_k+1} > x' - r > (x' + x)/2$$

and

$$y_{n_k+1} > y' - r > (x' + x)/2.$$

It follows:

$$x_n > (x' + x)/2 > x,\ \forall n > K$$

which is inconsistent with the hypothesis that: $\lim_{k\to\infty} x_{n_k} = x$. If $x \leqslant y' \leqslant x' < y$ we obtain a similar contradiction by choosing: $0 < r < (y - x')/2$.

Analogously we can prove that:

(15) $$y' = x \text{ or } y' = y$$

hold.

If M satisfies (12) and (13), from (14) and the continuity of M we get:

$$x = M(x, y) \text{ or } y = M(x, y)$$

thus $x = y$. If M' satisfies (12) and (13) we use (15).

The hypothesis $x > y$ gives analogously $x = y$. Hence:

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k} = x$$

which leads to:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x.$$

Remark. These results permit to compose even discontinuous means like:

$$M(x, y) = \begin{cases} \min(x, y) & \text{if } x \leqslant a \\ \max(x, y) & \text{if } x > a. \end{cases}$$

As concerns the result of the composition, we remind here some results. The most known is the arithmetic-geometric mean of Gauss:

$$\frac{1}{A\boxed{g}G(x_0, y_0)} = \frac{2}{\pi}\int_0^{\pi/2} (x_0^2 \cos^2 t + y_0^2 \sin^2 t)^{-1/2}\, dt.$$

It is also proved (see [13]) that:

$$A\boxed{a}G(x_0, y_0) = \begin{cases} \dfrac{(y_0^2 - x_0^2)^{1/2}}{\arccos(x_0/y_0)}, & x_0 < y_0 \\[2ex] \dfrac{(x_0^2 - y_0^2)^{1/2}}{\operatorname{arch}(x_0/y_0)}, & x_0 > y_0. \end{cases}$$

In [32] we proved that:

$A_p\boxed{a}A_q = A_r$, $G_p\boxed{a}G_q = G_r$, $H_p\boxed{a}H_q = H_r$ with $r = pq/(1 - p + pq)$ hold and

$A_p\boxed{g}A_q = A_s$, $G_p\boxed{g}G_q = G_s$, $H_p\boxed{g}H_q = H_r$, where $s = q/(1 - p + q)$

It is easy to see that:

$$A_p\boxed{g}H_{1-p} = G$$

and in [7] it is also remarked that:

$$P_r\boxed{g}P_{-r} = G.$$

Generally it is difficult to find the compound mean. For instance, G.D. Song made a conjecture for $A_p\boxed{g}G_p$ (see [23]) but it was invalidated by J. Wimp in [42].

As concerns the relation between the compound means, we have the result:

LEMMA. *If M and M' are increasing, then*:

$$M\boxed{g}M' \prec M\boxed{a}M'.$$

We can apply it to deduce relations among the above pairs of means. Of course, the most interesting result is:

$$A\boxed{g}G \prec A\boxed{a}G.$$

(Try to prove it directly!).

The rate of convergence of the sequences to the common limit was also studied for example in [15], [16], [17] and [33].

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