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ON DUALITY FOR GENERALIZED PSEUDOMONOTONIC PROGRAMMING

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1. Introduction. Pseudomonotonic and quasimonotonic programming has been extensively studied during last two decades. In [9] and [10] Martos shows that if the objective function is pseudomonotonic and the feasible set is a polyhedral convex set, then an optimal solution for this nonlinear programming problem, if it exists, can be found at a vertex of the feasible set and that the simplex procedure can be modified to solve such problems.

Kucher [8] presents another simplex procedure, which converges in a finite number of iterations when the objective quasimonotonic function is supposed to be an indefinite differentiable function. Bector et al [1], [2], [3], Bhatt [4], Mond [11] and Tigan [13], [15] give another procedures for solving such problems, which may be also used in the case of nonconvex feasible set.

Tigan [14] proposed a decomposition algorithm for quasimonotonic programing. Also, Tigan [16] studied certain quasimonotonic max-min problems with linked linear constraints.

Finally, Mond [11] considered a dual to the pseudomonotonic programming problem and obtained some weak and strong duality results.

On the other hand, Dantzig [5] developed a simplex type algorithm for generalized linear programming and Thuente [12] obtained some duality results for this problem.

The purpose of this paper is to develop some duality results for pseudomonotonic programming with generalized linear constraints.

2. Definitions and preliminaries. In this section we will briefly summarize some basic definitions and properties of the class of pseudomonotonic functions which are nonlinear and — beyond this — nonconcave.

DEFINITION 1. A function f from $D \subset \mathbb{R}^n$ into $\mathbb{R}(f : D \to \mathbb{R})$ is said to be *pseudo-convex* if

$$(y - x) \nabla f(x) \geqslant 0 \Rightarrow f(y) - f(x) \geqslant 0,$$

where ∇f is the gradient vector whose components are the partial derivatives of f and $\mathbb{D}$ is a convex set.

DEFINITION 2. The function f is *pseudo-concave* if $-f$ is pseudo-convex.

DEFINITION 3. A function f that is both pseudo-convex and pseudo-concave is called *pseudo-monotonic*.

Obviously, linear functions form the most important subclass of these, generally, nonlinear nonconcave (nonconvex) functions. A more general subclass of pseudomonotonic functions is that of the linear fractional functions with positive nominator on a given subset in $\mathbb{R}^n$ (cf. e.g. Martos [10]).

Other types of pseudomonotonic functions, which are not linear-fractional, were considered, for instance, by Mond [11], Tigan [15], Bector and Jolly [3].

Now, let X be a nonempty subset of D. We consider the following pseudomonotonic optimization problem.

PM. Maximize $f(x)$ subject to $x \in X$.

A useful linearization property, of the pseudomonotonic functions, obtained by Kortanek and Evans [7] in the particular case of the convex feasible set X, is the following:

THEOREM 1. (Tigan [15]) *Let f be a continuously differentiable pseudomonotonic function and the convex set D and let X be a closed bounded nonvoid subset of D. Then x' in X is an optimal solution of the problem PM if and only if x' is an optimal solution for the following linearized programming problem:*

$P(x')$. Maximize $\nabla f(x')x$, subject to $x \in X$.

The Theorem 2 below follows directly from the quasiconvexity property of the pseudomonotonic functions (see, for instance [6], P. 27 (ix), pp. 29–30).

THEOREM 2. *Let f be a pseudomonotonic function on D and let x', x'' be in X. If $\nabla f(x')\,x' < \nabla f(x')\,x''$ then $f(x') < f(x'')$.*

Theorems 1 and 2 suggest that maximizing a pseudomonotonic function over a closed bounded set is equivalent to maximizing some linear functions over the same set.

We mention that Theorems 1 and 2 are employed to justify the convergence (finite or infinite) of the linearization procedures given in [15] for pseudomonotonic programming with a nonconvex feasible set.

3. Dual problems. In the sequel, we consider the following pseudomonotonic programming problem with generalized linear constraints:

P. Maximize $f(x)$ subject to:

$$x_1a_1 + \ldots + x_na_n \leqslant b,\ x_j \geqslant 0,\ j \in I = \{1, 2, \ldots, n\}, \tag{1}$$

$$a_j \in K_j,\ j \in I, \tag{2}$$

where K_j $(j \in I)$ are convex bounded subsets in $\mathbb{R}^m$ and f is a pseudomonotonic function (see, Definition 3) over a convex set D (in $\mathbb{R}^n$) which includes the set X of all points $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, which verify the constraints (1) for some $a_j \in K_j (j \in I)$.

In problem P, x as well as $a_j (j \in I)$ are vectors of decision variables, so that, an optimal solution of problem P is a pair (x', A'), where $x' \in X$ and A' is the matrix having the columns a'_j $(j \in I)$ for which the optimum is achieved.

We associate to problem P the following dual problem with nonlinear constraints:

DP. Minimize $f(u)$ subject to

$$a_j y \geqslant \frac{\partial f(u)}{\partial x_j},\ \text{for any } a_j \in K_j,\ j \in I, \tag{3}$$

$$u\nabla f(u) \geqslant by,\ y \geqslant 0, \tag{4}$$

where $u \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are dual decision variable vectors.

The problem *DP* represents a generalization of Mond's [11] dual for usual pseudomonotonic programming.

THEOREM 3. (Weak duality) *If f is pseudomonotonic then for any feasible solution (x, A) of P and any feasible solution (y, u) of DP, the inequality $f(x) \leqslant f(u)$ holds.*

Proof. From (1) and $y \geqslant 0$, we have:

$$yAx \leqslant yb,\ \text{for some } a_j \in K_j\ (j \in I), \tag{5}$$

and from (3) and $x \geqslant 0$, it follows that:

$$x^tA^ty \geqslant x^t\nabla f(u),\ \text{for all } a_j \in K_j\ (j \in I). \tag{6}$$

But then, for some $a_j \in K_j$ $(j \in I)$, the constraints (5) and (6) imply:

$$yb \geqslant x^t\,\nabla f(u),$$

where, by (4), it results from:

$$u^t\nabla f(u) \geqslant yb \geqslant x^t\nabla f(u)$$

that:

$$(x - u)^t\,\nabla f(u) \leqslant 0. \tag{7}$$

Since f is pseudomonotonic, it is pseudoconvex, so that (7) implies the inequality $f(x) \leqslant f(u)$.

THEOREM 4. (*Strong duality*) *If x° is an optimal solution for P, then there exists $y \in \mathbb{R}^m$ such that (y, x°) is an optimal solution for DP and the optimal values of primal and dual problems are equal.*

Proof. By Theorem 1, x° is an optimal solution for P if and only if it is an optimal solution for the generalized linear programming problem :

P1. Find

$$\max \nabla f(x^\circ)\, x$$

subject to the generalized linear constraints (1) and (2).

Then, by duality theory for generalized linear programming due to Thuente [12], it follows that the dual of *P1* is :

DP1. Find

$$\min by$$

subject to :

$$(8)\qquad a_j y \geqslant -\frac{\partial f(x^\circ)}{\partial x_j}, \text{ for all } a_j \in K_j (j \in I),$$

$$(9)\qquad y \geqslant 0.$$

From Thuente's duality results [12], for every optimal solution y' of *DP1*, we have :

$$by' = \nabla f(x^\circ)\, x^\circ.$$

Therefore (y', x°) is a feasible solution for *DP*. But, by Theorem 3, since x° is a feasible solution for P (that is $x^\circ \in X$) it follows that :

$$f(x^\circ) \leqslant f(u)$$

for any feasible solution (y, u) of *DP*. It results that (y', x°) is an optimal solution for *DP*.

4. Linearization algorithm. In [3], [4] and [15] algorithms for solving optimization problems with pseudomonotonic objective functions are discussed. These approaches use nice linearization properties of the pseudomonotonic functions (see, Theorems 1 and 2) to end up, for convex polyhedral feasible set, at linear programs which can be solved employing simplicial algorithms.

The linearization alogorithm below envisages to find a finite sequence of feasible points, by solving a certain number of generalized linear programs. The last point $x' \in X$ in this sequence is a point for which Theorem 1 holds.

This algorithm represents an application of the linearization algorithm given in [15] to the pseudomonotonic programming with generalized linear constraints (problem *P*).

Algorithm.

Step 1. Choose a feasible point (x°, A°) for problem P and take $i = 0$.

Step 2. Solve the generalized linear program :

$P(x^i)$. Find

$$s_i = \max \nabla f(x^i)\, x$$

subject to the constraints (1) and (2).

Let (x^{i+1}, A^{i+1}) be an optimal solution of the problem $P(x^i)$, which is obtained by the generalized linear programming simplex algorithm [5].

Step 3. (i) If the inequality

$$\nabla f(x^i)\, x^i < s_i,$$

holds, then go to Step 2 with i replaced by $i + 1$.

(ii) If $\nabla f(x^i) x^i = s_i$, stop. By Theorem 1, (x^i, A^i) is an optimal solution for problem P.

We have the following result concerning the finite convergence of the linearization algorithm for generalized pseudomonotonic programming.

THEOREM 5. *If $K_j (j \in I)$ are convex nonvoid bounded polyhedral subsets in $\mathbb{R}^m$, then the linearization algorithm for generalized pseudomonotonic program P finishes after a finite number of iterations.*

Proof. Since the feasible set of problem P can be transformed into a convex polyhedral set having a finite number of extremal points, the theorem results easily by Theorem 7 from [15].

5. Conclusions. In this paper we considered a Mond type dual for pseudomonotonic programming with generalized linear constraints.

Two duality results are obtained. In order to prove the "strong duality theorem" a linearization result for pseudomonotonic programming and the duality property of generalized linear programming are used. The strong duality theorem shows that a pseudomonotonic program with nonlinear constraints, namely the dual problem *DP*, can be solved by means of the linearization algorithm (see, section 4) applied to the primal problem *P*.

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