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ON A BICRITERION MAX-MIN FRACTIONAL PROBLEM

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1. Introduction. In this paper we consider a max-min bicriterion fractional problem with separate linear constraints. We prove that this problem can be transformed into a bicriterion fractional maximization problem.

For this last type of nonlinear program, Warburton [15] described a finite procedure which consists in solving a one-parameter linear program and a series of one-dimensional maximizations.

Another approach to the general multiobjective linear-fractional programing was presented by Weber [16]. He considered an interactive procedure for the pseudomonotonic multiobjective programming, to obtain an efficient solution which satisfies some decision maker's preferences.

Our algorithm for the bicriterion max-min fractional problem consists in applying Warburton's procedure to solve a certain bicriterion fractional maximization program.

Concerning the bicriterion maximization problem, we mention that Geoffrion [6] studied a bicriterion mathematical program of the form:

$$Po. \qquad \max \{U(f_1(x), f_2(x)) : x \in S\},$$

where f_1 and f_2 are real-valued concave criterion functions of the n-vector x of decision variables that are constrained to lie in a convex subset S of R^n, and U is a real-valued increasing (i.e., monotone nondecreasing in each argument) ordinal utility indicator function defined on the set

$$f(S) = \{(f_1(x), f_2(x)) : x \in S\}.$$

He suggested a method for solving problem Po based on any known parametric programming algorithm for the parametric subproblem:

$$\max \{\alpha f_1(x) + (1 - \alpha) \; f_2(x) : x \in S\},$$

where the parameter α varies over the unit interval $(0 \leqslant \alpha \leqslant 1)$.

Warburton [15] describes also a procedure for solving problem Po having the objective functions f_1 and f_2 as linear fractional functions, S

a compact polyhedral set and U a real-valued utility function, nondecreasing in f_1 and f_2 and continuous on the set $f(S)$.

We mention that this problem does not always verify the concavity assumption for f_1 and f_2 considered by Geoffrion [6]. Hence, the fractional bicriterion Warburton's problem ordinarily is not a particular case of the problem considered by Geoffrion.

Applying some results from [6] and [3], Warburton showed how the problem Po can be reduced to a one-parameter linear program and a finite set of one-dimensional subproblems.

A special case of Po is also considered in [15]:

$$U(f_1(x), f_2(x)) = a' f_1(x) + a'' f_2(x), \tag{1.1}$$

where a' and a'' are real nonnegative numbers.

Ritter [8] gave an algorithm for the special case of (1.1) in which the objective is to maximize the sum of a linear and a linear-fractional function.

The purpose of this paper is to show how Warburton's procedure can be applied to solve a more general case of Po, namely, a bicriterion max-min fractional problem with non-joint polyhedral constraints. The plan of the paper is as follows. Section 2 contains the problem formulation and some general assumptions. The transformation of the max-min problem into usual maximization program is given in section 3. In the next section, an algorithm for solving the max-min fractional bicriterion problem is given.

2. Problem formulation. The max-min problem under consideration is of the form: P. Find

$$V = \max_{x \in X} \min_{y \in Y} h(F(x), Q(x, y)),$$

where $X \subset R^n$ and $Y \subset R^m$ are two compact convex polyhedral given sets, $h : D \to R (D \subset R^2)$ is a real-valued function nondecreasing and continuous on the set

$$D = \{(F(x), Q(x, y)) : x \in X, y \in Y\},$$

and $F : X \to R$, $Q : X \times Y \to R$ are fractional functions of the form:

$$F(x) = \frac{cx + p}{fx + q}, \quad x \in X, \tag{2.1}$$

$$Q(x, y) = \frac{xCy + ey + gx}{dx + r}, \quad x \in X, \ y \in Y, \tag{2.2}$$

This problem has an important application in modeling various conflict situations.

In (2.1) and (2.2), C is a given $n \times m$ real matrix, $e \in R^m$, $g, f, c \in R^n$ and p, q, r are given real numbers.

Without loss of generality, in what follows, we consider the feasible sets X and Y of the form:

$$X = \{x \in R^n : Ax \leq a, \ x \geq 0\}, \tag{2.3}$$

$$Y = \{y \in R^m : By \leq b, \ y \geq 0\}, \tag{2.4}$$

where A is an $s \times n$ real matrix, B is a $k \times m$ real matrix, $a \in R^s$ and $b \in R^k$.

DEFINITION 1.1 A point (x', y') in $X \times Y$ is said to be optimal solution for problem P, if:

i) $h(F(x'), Q(x', y')) = V$.

ii) $\min_{y \in Y} h(F(x'), Q(x', y)) = h(F(x'), Q(x', y'))$.

On functions F and Q, we make the following assumptions:

$$fx + q > 0, \quad \text{for all } x \text{ in } X. \tag{2.5}$$

$$dx + r > 0, \quad \text{for all } x \text{ in } X. \tag{2.6}$$

The problem P represents a generalization of the bicriterion linear fractional programming problem considered by Warburton [15], that is, the case when $C = 0$ and $e = 0$.

The particular case of problem P, when

$$h(F(x), Q(x, y)) = \max (F(x), Q(x, y)),$$

was studied by Belenkii [1]. Other approaches of max-min problems related to problem P have been realized in refs. [4], [5], [7], [9] – [14].

References [5], [9] and [13] deal with nonlinear fractional or quasi-monotonic max-min problems with linked constraints. Further extensions to max-min problems under separate linear constraints having various fractional objective functions have been studied by Cimoca and Tigan [4], Stancu-Minasian and Tigan [10], [11] and Tigan [12], [14]. Linear fractional max-min problems with disjunctive constraints have been considered by Patkar and Stancu-Minasian [7].

3. Main results. In what follows, we show that the max-min problem P can be transformed into a maximization problem of the form Po (problem $P1$ given below).

LEMMA 3.1 *If h is continuous and nondecreasing on D, then:*

$$\max_{x \in X} \min_{y \in Y} h(F(x), Q(x, y)) = \max_{x \in X} h(F(x), \min_{y \in Y} Q(x, y)).$$

Proof. Denote

$$G(x) = \min_{y \in Y} Q(x, y), \text{ for all } x \text{ in } X.$$

Since h is a continuous and nondecreasing function on D, for every x in X, we have:

$$\min_{y\in Y} h(F(x), Q(x, y)) = h(F(x), \min_{y\in Y} Q(x, y)),$$

so, it follows that:

$$\max_{x\in X} \min_{y\in Y} h(F(x), Q(x, y)) = \max_{x\in X} h(F(x), G(x)).$$

Consequently, by Lemma 3.1, the problem P can be reduced to the following bicriterion maximization problem:

P1. Find

$$V1 = \max_{x\in X} h(F(x), G(x)). \tag{3.1}$$

Obviously, $V = V1$ and if (x', y') is an optimal solution for P, then x' is an optimal solution for *P1*.

In dealing with problem *P1* we can consider the solution for the bicriterion Pareto problem with the objective functions F and G over the feasible set X.

DEFINITION 3.1 A point x'' in X is said to be Pareto optimal, if $x' \in X$ and $F(x') \geqslant F(x'')$, $G(x') \geqslant G(x'')$ implies that $F(x') = F(x'')$ and $G(x') = G(x'')$.

Let E denote the set of Pareto optimal points in X.

The problem *P1* is closely related to the following problem:

P2. Find

$$V2 = \max_{(z,t)\in Z} h\left(\frac{cz + pt}{fz + qt}, \min_{y\in Y} (zCy + tey + gz)\right)$$

where

$$Z = \{(z, t) \in R^{n+1} : Az - at \leqslant 0,\ dz + rt = 1,\ z \geqslant 0,\ t \geqslant 0\}.$$

THEOREM 3.1 (i) *If $(z,t) \in Z$, then $t > 0$.*

(ii) *If (z', t') is an optimal solution for P2, then $x' = z'/t'$ is an optimal solution for P1.*

(iii) *If x' is an optimal solution for problem P1, then $\left(\frac{x'}{dx'+r}, \frac{1}{dx'+r}\right)$ is an optimal solution for P2.*

(iv) *The optimal values of problems P1 and P2 are equal, i.e. $V1 = V2$.*

Proof. The assertion (i) can be easily obtained by a similar argument as those used by Charnes-Cooper [2] for usual linear fractional programming. The properties (ii), (iii) and (iv) can be easily proved, by performing the variable change $z = tx$ in problem *P1*. We mention that by this change of variable the problem *P1* is transformed into the equivalent problem *P2*.

Now we consider the following problem related to problem *P2*:

P3. Find

$$V3 = \max_{z,t,u} h\left(\frac{cz + pt}{fz + qt}, gz - bu\right) \tag{3.2}$$

subject to:

$$Az - at \leqslant 0, \tag{3.3}$$

$$dz + rt = 1, \tag{3.4}$$

$$-uB \leqslant zC + te, \tag{3.5}$$

$$z \geqslant 0,\ t \geqslant 0,\ u \geqslant 0. \tag{3.6}$$

Denote by T the feasible set of problem *P3*, i.e., the set of points (z, t, u) in R^{n+k+1}, which verify the constraints (3.3) — (3.6).

THEOREM 3.2. *If h is continuous and nondecreasing on D, then:*

(i) *problems P2 and P3 have the same optimal values, that is $V2 = V3$;*

(ii) *if (z', t', u') is an optimal solution for P3, then (z', t') is an optimal solution for P2;*

(iii) *if (z', t') is a optimal solution for P2, then there exists $u' \geqslant 0$, such that (z', t', u') is an optimal solution for P3.*

Proof. Indeed, for every $(z, t) \in Z$, the problem

$$M(z, t) = \min_{y\in Y} (zCy + tey + gz)$$

is a linear program. Since Y is a nonempty compact set of the form (2.4), by the duality property of linear programming, it follows that:

$$M(z, t) = \max_u \{gz - bu : -uB \leqslant zC + te,\ u \geqslant 0\}.$$

Consequently, for every (z, t) in Z, we have:

$$h\left(\frac{cz + pt}{fz + qt}, \min_{y\in Y} (zCy + tey + gz)\right) = h\left(\frac{cz + pt}{fz + qt}, \max_{u\in S(z,t)} (gz - bu)\right) \tag{3.7}$$

where:

$$S(z, t) = \{u \in R^k : -uB \leqslant zC + te,\ u \geqslant 0\},$$ for all (z, t) in Z.

Since h is continuous and nondecreasing, it results that:

$$h\left(\frac{cz + pt}{fz + qt}, \max_{u\in S(z,t)} (gz - bu)\right) = \max_{u\in S(z,t)} h\left(\frac{cz + pt}{fz + qt}, gz - bu\right), \tag{3.8}$$

for all (z, t) in Z.

From (3.7) and (3.8), it follows immediately that the properties (i), (ii) and (iii) of the theorem hold.

Therefore, by Lemma 3.1 and Theorems 3.1 and 3.2, it follows that the max-min problem P can be reduced to a bicriterion linear fractional problem *P3* with linear constraints, to which Warburton's parametrical procedure [15] can be applied.

4. The algorithm. In this section, we described an algorithm for solving the problem P.

Under assumptions (2.5) and (2.6) and because X and Y are nonempty compact sets, the following values are well-defined:

$$f_1' = \max \left\{ \frac{cz + pt}{fz + qt};\ (z, t) \in Z \right\}, \tag{4.1}$$

$$f_2' = \max \{gz - bu : (z, t, u) \in T\}, \tag{4.2}$$

$$f_1'' = \max \left\{ \frac{cz + pt}{fz + qt} : (z, t) \in Z,\ gz - bu \geq f_2' \right\}, \tag{4.3}$$

$$f_2'' = \max \left\{ gz - bu : (z, t, u) \in T,\ \frac{cz + pt}{fz + qt} \geq f_1' \right\}. \tag{4.4}$$

Denote by E' the Pareto optimal solutions over T.

From (4.1) – (4.3), it is obvious that:

$$f_1'' \leq \frac{cz + pt}{fz + qt} \leq f_1'$$

whenever $(z, t, u) \in E'$.

For each w in the interval $[f_1'', f_1']$, consider the problem: $P(w)$. Find

$$V(w) = \max \left\{ gz - bu : (z, t, u) \in T,\ \frac{cz + pt}{fz + qt} \geq w \right\}$$

Further the following results are useful to justify the algorithm.

LEMMA 4.1 (/3/, /15/) *If $(z', t', u') \in E'$, then there exists a scalar $w \in [f_1'', f_1']$ such that (z', t', u') solves $P(w)$. Conversely, each optimal solution of problem $P(w)$ for $w \in [f_1'', f_1']$ is Pareto optimal.*

LEMMA 4.2 (/6/) *If the assumptions of Lemma 3.1 hold, then there exists an optimal solution (z', t', u'), of problem P3, such that it is Pareto optimal.*

From Lemmas 4.1 and 4.2, it follows that:

LEMMA 4.3 *If the assumptions of Lemma 3.1 hold, then an optimal solution of P3 may be found among the solutions of $P(w)$ over the interval $f_1'' \leq w \leq f_1'$.*

Since (2.5) holds, the problem $P(w)$ may be rewritten: $P(w)$. Find

$$V(w) = \max \{gz - bu : (z, t, u) \in T,\ cz + pt \geq w(fz + qt)\}.$$

Then, the solution of $P(w)$ for $f_1'' \leq w \leq f_1'$ according to Warburton's procedure [15], leads to the following algorithm for solving problem P.

Step 1. Solve the problems (4.1), (4.2) and (4.3) to obtain f_1', f_1'' and f_2'.

Step 2. Apply row parametric procedure to the linear program $P(w)$ for $f_1'' \leq w \leq f_1'$, to obtain critical values

$$f_1'' = w_0 \leq w_1 \leq \ldots \leq w_{r'} = f_1'$$

and the correspoding points $Z_i' = (z^i, t^i, u^i)$ solving $P(w_i)$.

Step. 3. Solve r' one-dimensional subproblems defined by: Q_i. Find

$$W_i = \max \{h(H(\bar{Z})) : \bar{Z} \in [Z_{i-1}, Z_i]\},$$

for $1 \leq i \leq r'$, where by $\bar{Z} = (\bar{z}, \bar{t}, \bar{u})$ we denote a current feasible solution of $P3$ and by $H : T \to R^2$ the function:

$$H(z, t, u) = \left(\frac{cz + pt}{fz + qt},\ gz - bu \right),\ \text{for all } (z, t, u) \text{ in } T.$$

Let Z_i', for $1 \leq i \leq r'$, be the corresponding optimal solution of Q_i.

Step 4. Find $j \in \{1, 2, \ldots, r'\}$ such that:

$$h(H(Z_j')) = \max \{W_i : 1 \leq i \leq r'\}.$$

Then $Z' = Z_j'$ is an optimal solution for $P3$. Let $Z' = (z', t', u')$ denote this optimal solution.

Step 5. Take $x' = z'/t'$, which, according to Theorem 3.1, is an optimal solution for problem $P1$.

Step 6. Solve the following linear program:

$$\min_{y \in Y} \frac{x'Cy + ey + gx'}{dx' + r}.$$

Let y' be an optimal solution for this linear program. Then (x', y') is an optimal solution for problem, P, and the algorithm stops.

As is known, by Charnes-Cooper transformation [2], problems (4.1) and (4.3) in Step 1 are equivalent to linear programs. In Step 2, w_i occurs at the i-th basis change in the parametric solution of $P(w)$. The Step 3 is justified by Theorem 1 from [15] and Lemma 4.3, while Step 4 is waranted by Theorem 3.1 and Step 5 by Lemma 3.1 (see also Definitions 1.1 and 3.1).

Since h is a nondecreasing function, it results that for every (z', t', u') in E', we have $x' = z'/t'$ in E and conversely.

5. Conclusions. In this paper we presented an application of Warburton's parametric procedure for solving the max-min fractional problem P.

Although problem P is nonlinear and nonconvex, it is possible to obtain an optimal solution, and hence a Pareto optimal point for the corresponding bicriterion problem (see, Definition 3.1) by a rather simple algorithm. This algorithm uses only one-dimensional parametric linear programming techniques and one-dimensional nonlinear maximization procedures. Moreover, in the initial and final steps only linear fractional or linear programs must be solved.

Another bicriterion max-min fractional problem, related to problem P, is the following problem :

P'. Find

$$V' = \max_{x \in X} \min_{y \in Y} h\left(\frac{cx + p}{fx + q}, \frac{xCy + ey + gx}{\alpha y + \beta}\right)$$

where h, c, p, f, q, C, e, g has the same meaning as in the problem P and $\alpha \in R^m$, $\alpha \in R$.

A similar parametrical approach to those for problem P can be used for solving the problem P'.

We mention also that Fujishige, Katoh and Ichimori [17] have been considered a somewhat similar min-max problem with discrete submodular constraints.

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