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ON THE ABSOLUTE CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper a theorem of Mishra and Srivastava [3] has been proved under weaker conditions.

1. Introduction. Let $\sum a_n$ be a given infinite series with partial sums s_n, and $w_n = na_n$. By u_n and t_n we denote the n-th $(C, 1)$ means of the sequences (s_n) and (w_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geqslant 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} | u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

But since $t_n = n(u_n - u_{n-1})$ (see [2]), condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1.2}$$

Mishra and Srivastava [3] have proved the following theorem for $|C,1|_k$ summability factors of infinite series.

THEOREM A. *Let (X_n) be a positive non-decreasing sequence and there be sequences (λ_n) and (β_n) such that*

$$|\Delta\lambda_n| \leqslant \beta_n \tag{1.3}$$

$$\beta_n \to 0 \text{ as } n \to \infty \tag{1.4}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,X_n < \infty \tag{1.5}$$

$$|\lambda_n|\,X_n = O(1) \text{ as } n \to \infty. \tag{1.6}$$

If

$$\sum_{\nu=1}^{n} \frac{1}{\nu} |s_\nu|^k = O(X_n) \text{ as } n \to \infty, \tag{1.7}$$

then the series $\Sigma a_n \lambda_n$ is summable $|C, 1|_k$, $k \geqslant 1$.

2. The aim of this paper is to prove Theorem A under weaker conditions. Now, we shall prove the following theorem.

THEOREM. *Let (X_n) be a positive non-decreasing sequence and the sequences (λ_n) and (β_n) are such that conditions (1.3) — (1.6) of Theorem A are satisfied. If*

$$\sum_{v=1}^{n} \frac{1}{v} |t_v|^k = O(X_n) \text{ as } n \to \infty, \tag{2.1}$$

then the series $\sum a_n \lambda_n$ is summable $|C, 1|_k$, $k \geqslant 1$.

REMARK. It should be noted that the condition (1.7) implies the condition (2.1), the converse need not be true. In fact, since

$$u_n = \frac{1}{n+1} \sum_{v=0}^{n} s_v, \tag{2.2}$$

we have

$$(n+1)\, u_n - n u_{n-1} = s_n.$$

Thus, using the fact that $n(u_n - u_{n-1}) = t_n$, we have $t_n = s_n - u_n$. We can now write

$$\sum_{v=1}^{n} \frac{1}{v} |t_v|^k \leqslant 2^k \left\{ \sum_{v=1}^{n} \frac{1}{v} |s_v|^k + \sum_{v=1}^{n} \frac{1}{v} |u_v|^k \right\}. \tag{2.3}$$

On the other hand, using (2.2) and applying Hölder's inequality, we get

$$\sum_{v=1}^{n} \frac{1}{v} |u_v|^k = \sum_{v=1}^{n} \frac{1}{v} \left| \frac{1}{v+1} \sum_{p=0}^{v} s_p \right|^k \leqslant 2^k \left\{ \sum_{v=1}^{n} \frac{1}{v^{k+1}} \left\{ \sum_{p=1}^{v} |s_p| \right\}^k \right.$$

$$\left. + |s_0|^k \sum_{v=1}^{n} \frac{1}{v^{k+1}} \right\} \leqslant 2^k \sum_{v=1}^{n} \frac{1}{v^2} \left\{ \sum_{p=1}^{v} |s_p|^k \right\} \times \left\{ \frac{1}{v} \sum_{p=1}^{v} 1 \right\}^{k-1}$$

$$+ 2^k |s_0|^k \sum_{v=1}^{n} \frac{1}{v^{k+1}} = O(1) \sum_{p=1}^{n} |s_p|^k \sum_{v=1}^{n} \frac{1}{v^2} + O(1) = O(1) \sum_{p=1}^{n} \frac{1}{p} |s_p|^k + O(1),$$

so that, if (1.7) holds, then

$$\sum_{v=1}^{n} \frac{1}{v} |t_v|^k = O(X_n) \text{ as } n \to \infty, \text{ by (2.3)}.$$

To show the converse of this implication, it is sufficient to take $X_n = \log n$ and $a_n = \dfrac{1}{n}$, where $n > 1$. That is, if we choose (X_n) and a_n as above, then condition (2.1) holds but condition (1.7) does not hold. So we are weakening the hypotheses replacing (1.7) by (2.1).

3. We need the following lemma for the proof of our theorem.

LEMMA. ([3]). *Under the conditions on (X_n) and (β_n) as taken in the statement of Theorem A, the following conditions hold, when (1.5) is satisfied*

$$n \beta_n X_n = O(1) \text{ as } n \to \infty \tag{3.1}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.2}$$

4. Proof of the theorem. Let (T_n) be the n-th $(C, 1)$ mean of the sequence $(n a_n \lambda_n)$. That is to say that

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v. \tag{4.1}$$

To prove the theorem, we have to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |T_n|^k < \infty, \text{ by (1.2)}. \tag{4.2}$$

Applying Abel's transformation to the sum (4.1), we get that

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n-1} (v+1)\, t_v \Delta\lambda_v + t_n \lambda_n = T_{n,1} + T_{n,2}, \text{ say}.$$

To complete the proof of the theorem, by Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}|^k < \infty, \quad \text{for } r = 1,2. \tag{4.3}$$

Firstly, using (1.3) and applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}|^k = O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{ \sum_{v=1}^{n-1} v \beta_v |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v \beta_v |t_v|^k \right\} \times \left\{ \frac{1}{n} \sum_{v=1}^{n-1} v \beta_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v \beta_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^2} = O(1) \sum_{v=1}^{m} v \beta_v v^{-1} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{p=1}^{v} \frac{1}{p} |t_p|^k + O(1)\, m \beta_m \sum_{v=1}^{m} \frac{1}{v} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1)\, m\beta_m X_m = O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| X_v$$

$$+ O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| X_v + O(1)\, m\beta_m X_m = O(1) \text{ as } m \to \infty.$$

by virtue of (1.5), (2.1), (3.1), and (3.2).

It may be remarked that from the hypotheses of the theorem, (λ_n) is bounded. This can be shown like this. Since (X_n) is non-decreasing, $X_n \geqslant X_0$, which is a positive constant. Hence (1.6) implies that (λ_n) is bounded. Thus

$$\sum_{n=1}^{m} \frac{1}{n} |T_{n,2}|^k = \sum_{n=1}^{m} |\lambda_n| \, |\lambda_n|^{k-1} n^{-1} |t_n|^k = O(1) \sum_{n=1}^{m} |\lambda_n| \, n^{-1} |t_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m = O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1)$$

as $m \to \infty$, in view of (1.3), (1.6), (2.1), and (3.2).

Therefore, we get that

$$\sum_{n=1}^{m} \frac{1}{n} |T_{n,r}|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1,2.$$

This completes the proof of the theorem.

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