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APPROXIMATION OF CONTINUOUS SET-VALUED FUNCTIONS IN FRÉCHET SPACES. I†

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We consider set-valued functions defined on a compact Hausdorff topological space and with compact convex values in a Fréchet space; we give some conditions which ensure that a set of set-valued functions is a Korovkin set with respect to equicontinuous nets of monotone linear operators.

Introduction. In some recent papers [2], [3], [4] and [6] the interest of Korovkin-type approximation theory has been extended to linear operators on set-valued function cones, motivated by the variety of circumstances in which set-valued functions are involved, such as optimal control theory, mathematical economics and probability theory.

In a precursory study of Vitale [6], the approximation of Hausdorff continuous set-valued functions with compact convex values in a finite dimensional normed space has been discussed by means of Bernstein polynomials on the compact real interval [0,1].

Afterwards, Keimel and Roth have established in [3] a Korovkin-type theorem for set-valued Hausdorff continuous functions by means of Korovkin positive systems for single-valued real functions; the same theorem has been improved in [2] by using suitable upper and lower envelopes.

Finally, Keimel and Roth in [4] have developed an abstract formulation of some locally convex topologies on ordered cones by introducing a notion of (upper, lower) continuity which constitutes the substitute of the Hausdorff continuity in normed spaces; their results generalize approximation processes to the case of infinite-dimensional locally convex topological spaces.

In this paper we are interested in the Korovkin-type approximation of set-valued functions with compact convex values in Fréchet spaces; this setting is not the most general as considered in [4] but provides us some tools of selection theory which will play a crucial rôle in achieving the main result. Moreover, the Korovkin-type theorem obtained is not expressed in terms of Korovkin systems for single-valued real functions as in [4].

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In the second part of this paper, we shall consider single-valued continuous functions with values in a Fréchet space and we shall study the convergence of equicontinuous nets of operators satisfying suitable conditions; as a consequence, we obtain a natural generalization of the well-known case of real valued functions.

1. Preliminaries and notation. In this Section we recall some preliminary definitions and some classical results on selection theory which will be used in the sequel.

All the vector spaces under consideration have to be considered over the field $\mathbb{R}$ of real numbers.

We begin by fixing a locally convex Hausdorff topological space E and a base $\mathfrak{B}$ of convex open neighborhoods of 0 in E; moreover, we denote by $\mathscr{C}onv\,(E)$ the cone of all non empty convex subsets of E, endowed with the natural addition and multiplication by positive scalars and by $\mathscr{C}\mathscr{C}onv(E)$ the set of all non empty compact convex subsets of E.

Let X be a topological space; if $f:X \to \mathscr{C}onv(E)$ is a set-valued function on X, we recall that a (continuous) selection of f is a (continuous) function $\varphi:X \to E$ satisfying $\varphi(x) \in f(x)$ for all $x \in X$.

Moreover, the following notations will be useful in the sequel; $\mathscr{C}(X, E)$ denotes the space of all continuous functions on X with values in E endowed with the topology of the uniform convergence; if $f_1,\ldots,f_n$ are set-valued functions on X, we consider the set-valued function $\mathrm{co}(f_1,\ldots,f_n)$ defined by putting, for each $x \in X$, $\mathrm{co}\,(f_1,\ldots,f_n)(x) = \mathrm{co}\,(f_1(x),\ldots,f_n(x))$, where $\mathrm{co}(f_1(x),\ldots,f_n(x))$ denotes the convex hull of $f_1(x) \cup \ldots \cup f_n(x)$; moreover, if $\varphi \in \mathscr{C}(X, E)$, $\{\varphi\}$ denotes the set-valued function defined by putting, for each $x \in X$, $\{\varphi\}(x) = \{\varphi(x)\}$ and finally, if $\varphi_1,\ldots,\varphi_n$ are in $\mathscr{C}(X, E)$, we simply write $\mathrm{co}(\varphi_1,\ldots,\varphi_n)$ instead of $\mathrm{co}\,(\{\varphi_1\},\ldots,\{\varphi_n\})$.

The following preliminary result is well-known and can be easily derived from [5, Lemma 4.1 and Theorem 3.2''].

PROPOSITION 1.1. *If X is a paracompact Hausdorff space and E is a Fréchet space, then each lower semicontinuous set-valued function admits a continuous selection.*

Moreover, if $f:X \to \mathscr{C}\mathscr{C}onv(E)$ is a lower semicontinuous set-valued function and if A is a closed subset of X and $\chi:A \to E$ is a continuous selection of $f_{|A}$, then χ can be extended to a continuous selection of f; in particular, if $x_0 \in X$ and $y_0 \in f(x_0)$, there exists a continuous selection of f which takes the value y_0 at x_0. ∎

Let X be a paracompact Hausdorff space and E a Fréchet space. If $f:X \to \mathscr{C}\mathscr{C}onv(E)$ is a lower semicontinuous set-valued function, in the sequel we shall denote by $\mathscr{S}el(f)$ the (non empty) convex set of all continuous selections of f; by virtue of the preceding Proposition 1.1 we have, for each $x \in X$,

$$f(x) = \bigcup_{\varphi \in \mathscr{S}el(f)} \{\varphi(x)\}. \tag{1.1}$$

Moreover, the following lemma will be useful in the following Sections.

LEMMA 1.2. *Let X be a paracompact Hausdorff space and E a Fréchet space. If $f, g:X \to \mathscr{C}\mathscr{C}onv(E)$ are lower semicontinuous set-valued function and $\lambda \in \mathbb{R}$, it results*

$$\mathscr{S}el(f+g) = \mathscr{S}el(f) + \mathscr{S}el(g), \tag{1.2}$$

$$\mathscr{S}el(\lambda f) = \lambda \mathscr{S}el(f). \tag{1.3}$$

Proof. We have only to show the inclusion $\mathscr{S}el(f+g) \subset \mathscr{S}el(f) + \mathscr{S}el(g)$, the other being trivial. To this end, fix $\varphi \in \mathscr{S}el(f+g)$ and let $(V_n)_{n\in\mathbb{N}}$ be a sequence of convex closed sets in E which is a base of neighborhoods of 0 satisfying $V_{n+1} \subset 2^{-n}V_n$ for each $n \in \mathbb{N}$. By induction, we show the existence of a sequence $(\psi_n)_{n\in\mathbb{N}}$ of continuous selections of f and a sequence $(\chi_n)_{n\in\mathbb{N}}$ of continuous selections of g such that, for each $x \in X$,

$$\psi_{n+1}(x) \in \psi_n(x) + V_n, \quad \chi_{n+1}(x) \in \chi_n(x) + V_n \tag{1.4}$$

and

$$\varphi(x) - \psi_n(x) - \chi_n(x) \in V_n. \tag{1.5}$$

Put $f_0 = f$ and $g_0 = g$; if $x \in X$, we have $\varphi(x) \in f(x) + g(x)$ and therefore there exist $y \in f(x)$ and $z \in g(x)$ such that $\varphi(x) = y + z$; by Proposition 1.1 there exist $\psi_x \in \mathscr{S}el\,(f)$ and $\chi_x \in \mathscr{S}el(g)$ satisfying $\psi_x(x) = y$ and $\chi_x\,(x) = z$; it follows the existence of an open neighborhood $N(x)$ of x such that $\varphi(t) - \psi_x(t) - \chi_x(t) \in V_0$ for each $t \in N(x)$. Since X is paracompact there exists a locally finite continuous partition of unity $(p_x)_{x\in X}$ on X subordinated to $(N(x))_{x\in X}$. For each $x \in X$, let $I(x)$ be the finite set of all $t \in X$ such that $p_t(x) > 0$. Then the functions $\psi_0:X \to E$ and $\chi_0:X \to E$ defined by putting

$$\psi_0(x) = \sum_{t\in I(x)} p_t(x)\,\psi_t(x), \quad \chi_0(x) = \sum_{t\in I(x)} p_t(x)\,\chi_t(x)$$

for each $x \in X$, are continuous selections of f and respectively g. Moreover ψ_0 and χ_0 satisfy condition (1.5). Now, suppose that ψ_n and χ_n satisfy conditions (1.4) and (1.5) and consider the set-valued functions $f_{n+1}:X \to \mathscr{C}\mathscr{C}onv(E)$ and $g_{n+1}:X \to \mathscr{C}\mathscr{C}onv(E)$ defined by putting, for each $x \in X$, $f_{n+1}(x) = f(x) \cap (\psi_n(x) + V_n)$ and $g_{n+1}(x) = g(x) \cap (\chi_n(x) + V_n)$; by virtue of [5, Propositions 2.3 and 2.5], f_{n+1} and g_{n+1} are lower semicontinuous and consequently we can apply the preceding argument to show the existence of continuous selections ψ_{n+1} of f_{n+1} and χ_{n+1} of g_{n+1} satisfying $\varphi(x) - \psi_{n+1}(y) - \chi_{n+1}(y) \in V_{n+1}$ for each $x \in X$. By the definitions of f_{n+1} and g_{n+1}, ψ_{n+1} and χ_{n+1} satisfy also condition (1.4).

By (1.4), the sequences $(\psi_n)_{n\in\mathbb{N}}$ and $(\chi_n)_{n\in\mathbb{N}}$ satisfy the Cauchy condition with respect to the uniform topology in $\mathscr{C}(X, E)$ and therefore they uniformly converge to continuous functions $\psi: X \to E$ and respectively $\chi:X \to E$; it follows that ψ and χ are continuous selections of f and respectively g and $\varphi = \psi + \chi$(cf. (1.5)).

In the following Section, we shall consider set-valued functions both lower and upper semicontinuous and we shall indicate these set-

-valued functions simply as continuous functions. The cone of all continuous set-valued functions from a topological space X in $\mathcal{CC}onv(E)$ will be denoted by $\mathcal{C}(X, \mathcal{CC}onv(E))$.

The cone $\mathcal{C}(X, \mathcal{CC}onv(E))$ is equipped with the topology of the uniform convergence; namely we shall say that a net $(f_\iota)_{\iota\in I}^{\leqslant}$ of set-valued functions from X to $\mathcal{CC}onv(E)$ converges to a set-valued function $f : X \to \mathcal{CC}onv(E)$, if, for each $V \in \mathfrak{B}$, there exists $\alpha \in I$ such that

$$f(x) \subset f_\iota(x) + V,\ f_\iota(x) \subset f(x) + V \text{ for each } x \in X \text{ and } \iota \geqslant \alpha.$$

In [4], Keimel and Roth have introduced the symmetric topology on $\mathcal{C}onv(E)$ by considering the family

$$(\{B \in \mathcal{C}onv(E) \mid B \subset A + V,\ A \subset B + V\})_{V\in\mathfrak{B}}$$

as a neighborhood base of an arbitrary element $A \in \mathcal{C}onv(E)$.

Since we shall restrict our attention to set-valued functions with values, in the subcone $\mathcal{CC}onv(E)$ of $\mathcal{C}onv(E)$, in this case the continuous set-valued functions coincide with the set-valued functions which are continuous with respect to the symmetric topology (cf., e.g., [1, Corollary 1, p. 67]); consequently, the notation $\mathcal{C}(X, \mathcal{CC}onv E))$ is consistent with that one used by Keimel and Roth and a set-valued function $f : X \to \mathcal{CC}onv(E)$ is continuous if and only if, for each $V \in \mathfrak{B}$ and $x_0 \in X$, there exists a neighborhood N of x_0 such that

$$(1.6) \qquad f(x) \subset f(x_0) + V, \quad f(x_0) \subset f(x) + V$$

for each $x \in N$.

We conclude this Section by recalling that in $\mathcal{C}(X, \mathcal{CC}onv(E))$ is defined the following order relation

$$(1.7) \qquad f \leqslant g \Leftrightarrow f(x) \subset g(x) \text{ for each } x \in X$$

for each set-valued functions $f, g\ X : \to \mathcal{CC}onv(E)$.

We shall also use the notation $f \leqslant g + V$ ($f,\ g \in \mathcal{C}(X, \mathcal{CC}onv(E))$ and $V \in \mathfrak{B}$) to indicate $f(x) \subset g(x) + V$ for each $x \in X$.

2. Approximation of continuous set-valued functions. In this Section we fix a compact Hausdorff topological space X and a Fréchet space E; we shall consider a subcone $\mathcal{C}$ of $\mathcal{C}(X, \mathcal{CC}onv(E))$ containing the single-valued functions (i.e. $\{\varphi\} \in \mathcal{C}$ for eah $\varphi \in \mathcal{C}((X, E))$ and we shall study the convergence of equicontinuous nets of monotone linear operators from $\mathcal{C}$ in $\mathcal{C}(X, \mathcal{CC}onv(E))$.

First of all, we recall that an operator $T : \mathcal{C} \to \mathcal{C}(X, \mathcal{CC}onv(E))$ from a subcone $\mathcal{C}$ of $\mathcal{C}(X, \mathcal{CC}onv(E))$ in $\mathcal{C}(X, \mathcal{CC}onv(E))$ is called linear if

$$T(f + g) = T(f) + T(g), \quad T(\lambda f) = \lambda T(f)$$

for each set-valued functions $f,\ g \in \mathcal{C}$ and $\lambda \geqslant 0$.

Observe that if $T : \mathcal{C} \to \mathcal{C}(X, \mathcal{CC}onv(E))$ is linear then $T(O) = 0$. Moreover, an operator $T : \mathcal{C} \to \mathcal{C}(X, \mathcal{CC}onv(E))$ is called monotone if

$$(2.1) \qquad f, g \in \mathcal{C},\ f \leqslant g \Rightarrow T(f) \leqslant T(g)$$

(cf. (1.7)).

Finally, we say that T is continuous if it is continuous with respect to the uniform symmetric topology induced on $\mathcal{C}$; thus, for each $V \in \mathfrak{B}$ there exists $U \in \mathfrak{B}$ such that, if $f,\ g \in \mathfrak{C}$ satisfy

$$f(x) \subset g(x) + U,\ g(x) \subset f(x) + U$$

for each $x \in X$ (i.e. $f \leqslant g + U$, $g \leqslant f + U$), then

$$T(f)(x) \subset T(g)(x) + V,\ T(g)(x) \subset T(f)(x) + V$$

for each $x \in X$ (i.e. $T(f) \leqslant T(g) + V$, $T(g) \leqslant T(f) + V$).

If K is a non empty compact convex subset of E, we denote by f_K the constant set-valued function on X of constant value K; we shall say that a subcone $\mathcal{C}$ of $\mathcal{C}(X, \mathcal{CC}onv(E))$ contains the constant set-valued functions if $f_K \in \mathcal{C}$ for each non empty compact convex subset K of E.

The following Lemma will be useful in the sequel.

LEMMA 2.1. *Let $\mathcal{C}$ be a subcone of $\mathcal{C}(X, \mathcal{CC}onv(E))$ containing the constant set-valued functions and let $(T_\iota)_{\iota\in I}^{\leqslant}$ be an equicontinuous net of monotone linear operators from $\mathcal{C}$ in $\mathcal{C}(X, \mathcal{CC}onv(E))$. Then, for each $V \in \mathfrak{B}$, there exists $U \in \mathfrak{B}$ such that*

$$f, g \in \mathcal{C},\ f \leqslant g + U \Rightarrow T_\iota(f) \leqslant T_\iota(g) + V \text{ for each } \iota \in I.$$

Proof. Let $V \in \mathfrak{B}$; since $(T_\iota)_{\iota\in I}^{\leqslant}$ is equicontinuous at 0, there exists $U_1 \in \mathfrak{B}$ such that, for each $f \in \mathcal{C}$.

$$f \leqslant U_1 \Rightarrow T_\iota(f) \leqslant V \text{ for each } \iota \in I.$$

Now, let $U \in \mathfrak{B}$ be such that $\bar{U} \subset U_1$ and let $f, g \in \mathcal{C}$ satisfy $f \leqslant g + U$; observe that the set $K = \bar{U} \cap (f(X) - g(X))$ is a non empty convex compact subset of E (cf., e.g. [1, Proposition 3, p. 42 and Theorem 1, p. 41]) and the constant set-valued function f_K satisfies the condition $f_K \leqslant U_1$; moreover, if $x \in X$ and if $y \in f(x)$, we have $y = z + u$ with $z \in g(x)$ and $u \in U$; it follows $u \in \bar{U} \cap (f(X) - g(X))$ and $y \in g(x) + f_K(x)$; since $x \in X$ and $y \in f(x)$ are arbitrary we have $f \leqslant g + f_K$; for each $\iota \in I$, the monotonicity of T_ι implies $T_\iota(f) \leqslant T_\iota(g + f_K) = T_\iota(g) + T_\iota(f_K) \leqslant T_\iota(g) + V$ and the proof is completed. ∎

In [4], Keimel and Roth have also introduced the class of uniformly continuous operators; and operator $T : \mathcal{C} \to \mathcal{C}(X, \mathcal{CC}onv(E))$ is called uniformly continuous (or briefly u-conti nuous) if, for each $V \in \mathfrak{B}$, there exists $U \in \mathfrak{B}$ such that, for each $f,\ g \in \mathcal{C}$,

$$f \leqslant g + U \Rightarrow T(f) \leqslant T(g) + V.$$

It is clear that an uniformly continuous operator is both continuous and monotone; as a consequence of Lemma 2.1, also the converse is true.

COROLLARY 2.2. *Let $\mathscr{C}$ be a subcone of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ containing the constant set-valued functions and let $T: \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E)) \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ be a linear operator of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ in itself.*
Then, the following statements are equivalent:
a) T is uniformly continuous;
b) T is continuous and monotone;
c) T is continuous at 0 and monotone.

Proof. The implications a) $\Rightarrow$ b) and b) $\Rightarrow$ c) are trivial and the implication c) $\Rightarrow$ a) follows from Lemma 2.1. ∎.

Thus, by virtue of the preceding Corollary 2.2, if $\mathscr{C}$ contains the constant set-valued functions, the study of the convergence of nets of monotone continuous linear operators from $\mathscr{C}$ in $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ is equivalent to the study of the convergence of nets of uniformly continuous operators.

In the sequel, we shall consider subcones $\mathscr{C}$ of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ containing the single-valued functions and monotone continuous linear operators $T: \mathscr{C} \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ satisfying the following conditions:

(2.2) for each $\varphi \in \mathscr{C}(X, E)$, $T(\{\varphi\})$ is single-valued;

(2.3) for each $f \in \mathscr{C}$ and $x \in X$,

$$T(f)(x) = \bigcup_{\varphi \in \mathscr{S}el(f)} T(\{\varphi\})(x).$$

By (1.1), the identity operator satisfies conditions (2.2) and (2.3).

In the second part we shall see that, in the case $E = \mathbb{R}$, monotone continuous linear operators from $\mathscr{C}(X, \mathbb{R})$ in itself generate in a natural way monotone continuous linear operators from $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ in itself satisfying (2.2) and (2.3).

Before stating the main result, we state the following fundamental definition which naturally arises from a similar one which is well-known for single-valued functions.

DEFINITION 2.3. *Let $\mathscr{C}$ be a subcone of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ containing the single-valued functions and let $T: \mathscr{C} \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ be a monotone continuous linear operator satisfying conditions (2.2) and (2.3).*

If H is a subset of $\mathscr{C}$, we shall say that H is a T-Korovkin set in $\mathscr{C}$ if, for each equicontinuous net $(T_\iota)^{\leq}_{\iota \in I}$ of monotone linear operators from $\mathscr{C}$ in $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ such that the net $(T_\iota(h))^{\leq}_{\iota \in I}$ converges to $T(h)$ for each $h \in H$, we also have that the net $(T_\iota(f))^{\leq}_{\iota \in I}$ converges to $T(f)$ for every $f \in \mathscr{C}$.

If T is the identity operator, a T-Korovkin set in $\mathscr{C}$ will be simply called a Korovkin set in $\mathscr{C}$.

As observed in [4], if H contains the constant set-valued functions, we can omit the equicontinuity of the net $(T_\iota)^{\leq}_{\iota \in I}$ of continuous monotone linear operators in Definition 2.3.

We have the following main theorem.

THEOREM 2.4. *Let X be a compact Hausdorff topological space, E a Fréchet space, $\mathscr{C}$ a subcone of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ containing the single-valued functions and $T: \mathscr{C} \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ a monotone continuous linear operator satisfying conditions (2.2) and (2.3).*

If H is a subset of $\mathscr{C}$ satisfying the following condition

(2.4) *for each $f \in \mathscr{C}$, $x_0 \in X$ and $V \in \mathfrak{B}$, there exists $h \in H$ such that*

$$f \leq h \text{ and } T(h)(x_0) \subset T(f)(x_0) + V,$$

then H is a T-Korovkin set in $\mathscr{C}$.

Proof. Let $(T_\iota)^{\leq}_{\iota \in I}$ be an equicontinuous net of monotone linear operators from $\mathscr{C}$ in $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ and suppose that the net $(T_\iota(h))^{\leq}_{\iota \in I}$ converges to $T(h)$ for each $h \in H$.

Fix $f \in \mathscr{C}$; in order to show that the net $(T_\iota(f))^{\leq}_{\iota \in I}$ converges to $T(f)$, we distinguish different cases.

1°) f is single-valued. Let $V \in \mathfrak{B}$ and consider $W \in \mathfrak{B}$ such that $3W - 3W \subset V$.

Let $x_0 \in X$ and consider $h \in H$ such that $f \leq h$ and $T(h)(x_0) \subset T(f)(x_0) + W$ (cf. (2.4)); by the continuity of $T(f)$ and $T(h)$, there exists an open neighborhood $N(x_0)$ of x_0 such that $T(h)(x) \subset T(f)(x) + 2W$ whenever $x \in N(x_0)$.

For each $\iota \in I$, T_ι is monotone and therefore $T_\iota(f) \leq T_\iota(h)$; on the other hand, the net $(T_\iota(h))^{\leq}_{\iota \in I}$ converges to $T(h)$ and therefore there exists $\alpha(x_0) \in I$ such that, for each $\iota \in I$, $\iota \geq \alpha(x_0)$,

$$(2.5) \qquad T_\iota(h) \leq T(h) + W, \quad T(h) \leq T_\iota(h) + W.$$

It follows, for each $\iota \in I$, $\iota \geq \alpha(x_0)$ and for each $x \in N(x_0)$, $T_\iota(f)(x) \subset T_\iota(h)(x) \subset T(h)(x) + W \subset T(f)(x) + 3W$.

Now, let x_0 vary in X and consider the open cover $(N(x_0))_{x_0 \in X}$ of X; since X is compact, there exists $x_1, \ldots, x_n \in X$ such that $X = N(x_1) \cup \ldots \cup N(x_n)$; let $\alpha \in I$ be such that $\alpha(x_i) \leq \alpha$ for each $i = 1, \ldots, n$. For each $\iota \in I$, $\iota \geq \alpha$, we have $T_\iota(f) \leq T(f) + 3W$; since $T(f)$ is single-valued (cf. (2.2)) this implies also $T(f) \leq T_\iota(f) - 3W$; therefore $T_\iota(f) \leq T(f) + V$ and $T(f) \leq T_\iota(f) + V$ and the proof is complete in this case.

2°) We consider the general case. Let $V \in \mathfrak{B}$ and consider $W \in \mathfrak{B}$ such that $3W \subset V$.

Let $x_0 \in V$; by (2.4), there exists $h \in H$ such that $f \leq h$ and $T(h)(x_0) \subset T(f)(x_0) + W$; moreover, the set $T(f)(x_0)$ is compact and therefore there exist $y_1, \ldots, y_n \in T(f)(x_0)$ such that $T(f)(x_0) \subset \bigcup_{i=1}^{n} \{y_i\} + W$; by (2.2) and (2.3), we may consider, for each $i = 1, \ldots, n$, a continuous selection φ_i of f such that $T(\{\varphi_i\})(x_0) = y_i$.

We have $\{\varphi_i\} \in \mathscr{C}$, $\{\varphi_i\} \leq f$ for each $i = 1, \ldots, n$ and further $T(f)(x_0) \subset \mathrm{co}\,(T(\{\varphi_1\}), \ldots, T(\{\varphi_n\}))(x_0) + W$; by virtue of the continuity of

$T(\{\varphi_1\}),\dots,T(\{\varphi_n\})$, $T(f)$ and $T(h)$ we may find an open neighborhood $N(x_0)$ of x_0 such that, for every $x\in N(x_0)$,

(2.6) $T(h)(x) \subset T(f)(x) + 2W$, $T(f)(x) \subset \mathrm{co}\,(T(\{\varphi_1\},\dots,T(\{\varphi_n\})(x) + 2W$.

At this point, we observe that $\{\varphi_i\}$ is single-valued for each $i=1,\dots,n$ and therefore the net $(T_\iota(\{\varphi_i\}))^{\leqslant}_{\iota\in I}$ converges to $T(\{\varphi_i\})$; moreover, since $h\in H$ the net $(T_\iota(h))^{\leqslant}_{\iota\in I}$ converges to h; hence, there exists $\alpha(x_0)\in I$ such that, for each $\iota\in I$, $\iota\geqslant\alpha(x_0)$ and $j=1,\dots,n$,

(2.7) $$T_\iota(\{\varphi_j\}) \leqslant T(\{\varphi_j\}) + W, \quad T(\{\varphi_j\}) \leqslant T_\iota(\{\varphi_j\}) + W,$$
$$T_\iota(h) \leqslant T(h) + W, \quad T(h) \leqslant T_\iota(h) + W.$$

By (2.6) and (2.7) we obtain, for each $\iota\in I$, $\iota\geqslant\alpha(x_0)$, $x\in N(x_0)$ and $j=1,\dots,n$,

$$T(\{\varphi_j\})(x) \subset T_\iota(\{\varphi_j\})(x) + W \subset T_\iota(f)(x) + W$$

and hence, since $T_\iota(f)(x)$ is convex,

(2.9) $T(f)(x) \subset \mathrm{co}(T(\{\varphi_1\},\dots,T(\{\varphi_n\})(x) + 2W \subset T_\iota(f)(x) + 3W \subset$
$$T_\iota(f)(x) + V;$$

on the other hand

(2.10) $T_\iota(f)(x) \subset T_\iota(h)(x) \subset T(h)(x) + W \subset T(f)(x) + 3W \subset T(f)(x) + V$.

Arguing on the compactness of X as in the first case, by (2.9) and (2.10) we deduce the existence of $\alpha\in I$ such that, for each $\iota\in I$, $\iota\geqslant\alpha$, $T(f)\leqslant T_\iota(f)+V$, $T_\iota(f)\leqslant T(f)+V$ and this completes the proof.

In the particular case where the operator T is the identity operator, we obtain the following Corollary.

COROLLARY 2.6. *Let X be a compact Hausdorff topological space, E a Fréchet space and $\mathscr{C}$ a subcone of $\mathscr{C}(X, \mathscr{CC}onv(E))$ containing the single-valued functions.*

If H is a subset of $\mathscr{C}$ satisfying the following condition

(2.11) *for each $f\in\mathscr{C}$, $x_0\in X$ and $V\in\mathfrak{B}$, there exists $h\in H$ such that*

$$f\leqslant h \text{ and } h(x_0)\subset f(x_0)+V,$$

then H is a Korovkin set in $\mathscr{C}$. ∎

REMARK 2.7 Under the hypotheses of Theorem 2.4, if the subcone $\mathscr{C}$ also contains the constant set-valued functions, condition (2.4) may be replaced by the following

(2.12) for each $f\in\mathscr{C}$, $x_0\in X$ and $V\in\mathfrak{B}$, there exists $h\in H$ such that

$$f\leqslant h+V \text{ and } T(h)(x_0)\subset T(f)(x_0)+V.$$

The proof is similar to that of Theorem 2.4 by using Lemma 2.1 instead of the monotonicity of T_ι in order to obtain (2.5) (condition (2.12) is applied to an element $U\in\mathfrak{B}$ contained in V and such that if $g\in\mathscr{C}(X, \mathscr{CC}onv(E))$ and if $f\leqslant g+U$, the $T_\iota(f)\subset T_\iota(g)+V$ for each $\iota\in I$).

Moreover, if we consider the identity operator condition (2.11) may be replaced by the following.

(2.13) for each $f\in\mathscr{C}$, $x_0\in X$ and $V\in\mathfrak{B}$, there exists $h\in H$ such that

$$f\leqslant h+V \text{ and } h(x_0)\subset f(x_0)+V.$$

This last condition constitutes a generalization of the result of Vitale [6]. Indeed, assume $X=[0,1]$, $E=\mathbb{R}^n$ $(n\geqslant 1)$, and consider the subspace Γ generated by the functions $e_i(x)=x^i (i=0,1,2$ and $x\in[0,1])$ and the set H of all set-valued functions $x\mapsto\gamma(x)\cdot\mathbb{B}$ $(x\in[0,1])$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$ and $\gamma\in\Gamma$. Then, we shall show that H satisfies condition (2.13) with $\mathscr{C}=\mathscr{C}(X, \mathscr{CC}onv(\mathbb{R}^n))$. Let $f\in\mathscr{C}(X, \mathscr{CC}onv(\mathbb{R}^n))$ and fix $x_0\in[0,1]$ and $\varepsilon>0$; since f is continuous at x_0, there exists $\delta>0$ such that, for each $x\in[0,1]\cap[x_0-\delta, x_0+\delta]$, $f(x)\subset f(x_0)+\varepsilon\cdot\mathbb{B}$. Moreover, there exist a compact convex set K in $\mathbb{R}^n$ such that $f(x)\subset K$ for each $x\in[0,1]$ (cf., e.g. [1, Proposition 3, p. 42 and Theorem 1, p. 41]) and a positive constant $M\in\mathbb{R}$ such that $K\subset f(x_0)+M\cdot\mathbb{B}$. Put $a=\max\{0, x_0-\delta\}$ and $b=\min\{1, x_0+\delta\}$ and consider a positive function $\gamma\in\Gamma$ such that $\gamma(x_0)=0$ and $\gamma\geqslant M$ in $[0,1]\setminus[a,b]$. Finally define the set valued function $h:[0,1]\to\mathscr{CC}onv(\mathbb{R}^n)$ by putting $h(x)=f(x_0)+(\gamma(x)+\varepsilon)\cdot\mathbb{B}$ for each $x\in[0,1]$. Then $h\in H$ and for each $x\in[a,b]$, $f(x)\subset f(x_0)+\varepsilon\cdot\mathbb{B}\subset f(x_0)+(\gamma(x)+\varepsilon\cdot\mathbb{B}=h(x)$, while, for each $x\in[0,1]\setminus[a,b]$, $f(x)\subset K\subset f(x_0)+M\cdot\mathbb{B}\subset f(x_0)+(\gamma(x)+\varepsilon)\cdot\mathbb{B}=h(x)$; hence $f\leqslant h$ and, since $h(x_0)=f(x_0)+\varepsilon\cdot\mathbb{B}$, the proof is complete. ∎

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