MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 20, N^{os} 1—2, 1991, pp. 25—38

APPROXIMATION OF CONTINUOUS SET-VALUED FUNCTIONS IN FRÉCHET SPACES. II†

MICHELE CAMPITI
(Bari)

Abstract. In this paper we apply some results obtained in the first part to the approximation of single-valued continuous functions defined on a compact Hausdorff topological space and with values in a Fréchet space.

Introduction. The well-known results on the Korovkin approximation of continuous real-valued functions have played a crucial rôle in the investigation of set-valued continuous functions (cf. [4], [5] and [8]) and the results obtained are strictly related to the existence of a Korovkin set of continuous single-valued functions; in the first part of this paper [2] we have followed a different approach to the approximation of set-valued continuous functions, which is independent of Korovkin sets of single-valued functions; this allows us to apply the results in the first part to the case of single-valued functions with values in a Fréchet space. As a consequence, we obtain a natural generalization of the well-known case of real valued functions.

We shall assume the same notation of the first part; E denotes a real Fréchet space, $\mathfrak{B}$ a base of convex open neighborhoods of 0 in E and $\mathscr{C}\mathscr{C}onv(E)$ is the cone of all non empty convex compact subsets of E.

Moreover, we fix a compact Hausdorff topological space X and we shall denote by $\mathscr{C}(X, E)$ the space of all continuous functions on X with values in E and by $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ the cone of all continuous set-valued functions from X in $\mathscr{C}\mathscr{C}onv(E)$; $\mathscr{C}(X, E)$ and $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ are both equipped with the topology of the uniform convergence.

Finally, we recall the notation $f \leqslant g + V$ ($f, g \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ and $V \in \mathfrak{B}$) to indicate $f(x) \subset g(x) + V$ for each $x \in X$.

Approximation of continuous vector-valued functions. In this Section, we shall apply the main theorem of the first part [2, Theorem 2.4] to obtain some Korovkin-type theorems for single-valued continuous functions.

We shall introduce a class $\mathscr{M}$ of linear continuous operators on the space $\mathscr{C}(X, E)$ which can be regarded as a generalization of monotone

---

† Work performed under the auspices of the G.N.A.F.A. (C.N.R.) and M.U.R.S.T (60%) and supported by I.N.d.A.M.

operators in an abstract setting and can be naturally extended to operators between set-valued continuous functions on a suitable subcone of $\mathscr{C}(X, \mathscr{C}\mathscr{C}_{onv}(E))$; this property allows us to study the operators in $\mathscr{M}$ by applying [2, Theorem 2.4] to the extended operators.

This program requires some preliminary results; in Lemma 1.1 we consider the subcone of $\mathscr{C}(X, \mathscr{C}\mathscr{C}_{onv}(E))$ to which we can extend the operators in the class $\mathscr{M}$; after we introduce the operators in $\mathscr{M}$ and study their basic properties, which are based on Proposition 1.2, together with some connections with monotone operators in the case $E = \mathbb{R}$ (Proposition 1.3); the mentioned extension property of the operators in $\mathscr{M}$ is given in Proposition 1.5; finally, in Theorem 1.7 and the subsequent Corollaries, we establish the main Korovkin type results.

Firstly, we need to consider the set

(1.1) $$\mathscr{F}(X, \mathscr{C}\mathscr{C}_{onv}(E)) = \{f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}_{onv}(E)) \mid \textit{there exist } \varphi_1, \ldots, \varphi_n \in \mathscr{C}(X, E) \textit{ such that } f(x) = \mathrm{co}(\varphi_1, \ldots, \varphi_n)(x) \textit{ for each } x \in X\}$$

(for each $x \in X$, $\mathrm{co}(\varphi_1, \ldots, \varphi_n)(x)$ denotes the convex hull of the set $\{\varphi_1(x), \ldots \varphi_n(x)\}$); in the following Lemma we list some properties of $\mathscr{F}(X, \mathscr{C}\mathscr{C}_{onv}(E))$.

LEMMA 1.1. *The set* $\mathscr{F}(X, \mathscr{C}\mathscr{C}_{onv}(E))$ *is a subcone of* $\mathscr{C}(X, \mathscr{C}\mathscr{C}_{onv}(E))$ *containing the single-valued functions.*

Moreover, if $f = \mathrm{co}(\varphi_1, \ldots, \varphi_n)$ *and* $g = \mathrm{co}(\psi_1, \ldots, \psi_m)$ *are in* $\mathscr{F}(X, \mathscr{C}\mathscr{C}_{onv}(E))$ *and if* $\lambda \geqslant 0$, *we have*

(1.2) $$f + g = \mathrm{co}(\varphi_1 + \psi_1, \ldots, \varphi_n + \psi_1, \ldots, \varphi_1 + \psi_m, \ldots, \varphi_n + \psi_m),$$

(1.3) $$\lambda f = \mathrm{co}(\lambda\varphi_1, \ldots, \lambda\varphi_n).$$

Proof. It is obvious that $\mathscr{F}(X, \mathscr{C}\mathscr{C}_{onv}(E))$ contains the single-valued functions; then, it suffices to show that (1.2) and (1.3) hold and from this it will follow that $\mathscr{F}(X, \mathscr{C}\mathscr{C}_{onv}(E))$ is a subcone of $\mathscr{C}(X, \mathscr{C}\mathscr{C}_{onv}(E))$.

Let $f = \mathrm{co}(\varphi_1, \ldots, \varphi_n)$ and $g = \mathrm{co}(\psi_1, \ldots, \psi_m)$ be in $\mathscr{F}(X, \mathscr{C}\mathscr{C}_{onv}(E))$ and fix $x \in X$; for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$, $\varphi_i(x) + \psi_j(x) \in \mathrm{co}(\varphi_1, \ldots, \varphi_n)(x) + \mathrm{co}(\psi_1, \ldots, \psi_m)(x)$ and consequently

$$\mathrm{co}(\varphi_1 + \psi_1, \ldots, \varphi_n + \psi_1, \ldots, \varphi_1 + \psi_m, \ldots, \varphi_n + \psi_m)(x) \subset \mathrm{co}(\varphi_1, \ldots, \varphi_n)(x) + \mathrm{co}(\psi_1, \ldots, \psi_m)(x).$$

Conversely, let $y = u + v$, with $u \in \mathrm{co}(\varphi_1, \ldots, \varphi_n)(x)$ and $v \in \mathrm{co}(\psi_1, \ldots, \psi_m)(x)$; then, there exist $\lambda_1, \ldots, \lambda_p \geqslant 0$ and $\mu_1, \ldots, \mu_q \geqslant 0$ such that $\sum_{i=1}^{p} \lambda_i = 1$, $\sum_{j=1}^{q} \mu_j = 1$ and $u = \sum_{i=1}^{p} \lambda_i \varphi_i(x)$, $v = \sum_{j=1}^{q} \mu_j \psi_j(x)$.

Consequently, we have

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j = \sum_{i=1}^{p} \lambda_i \sum_{j=1}^{q} \mu_j = 1$$

and

$$y = u + v = \sum_{i=1}^{p} \lambda_i \varphi_i(x) + \sum_{j=1}^{q} \mu_j \psi_j(x) = \sum_{i=1}^{p} \sum_{j=1}^{q} (\lambda_i \mu_j \varphi_i(x) + \lambda_i \mu_j \psi_j(x)) = \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j (\varphi_i + \psi_j)(x);$$

this yields $y \in \mathrm{co}(\varphi_1 + \psi_1, \ldots, \varphi_n + \psi_1, \ldots, \varphi_1 + \psi_m, \ldots, \varphi_n + \psi_m)(x)$.

Since $y \in \mathrm{co}(\varphi_1, \ldots, \varphi_n)(x) + \mathrm{co}(\psi_1, \ldots, \psi_m)(x)$ and $x \in X$ are arbitrary, (1.2) is true.

Finally (1.3) is trivial. ∎

If $n \geqslant 1$ and $\varphi_1, \ldots, \varphi_n \in \mathscr{C}(X, E)$, we shall consider the set

(1.4) $$A(\varphi_1, \ldots, \varphi_n) = \{\varphi \in \mathscr{C}(X, E) \mid \textit{for each } x \in X \textit{ there exists } i = 1, \ldots, n \textit{ such that } \varphi(x) = \varphi_i(x)\}$$

of all continuous functions on X whose graphs are contained in the union of the graphs of $\varphi_1, \ldots, \varphi_n$.

Observe that if $\varphi_1, \ldots, \varphi_n$ have not pairwise disjoint graphs, the set $A(\varphi_1, \ldots, \varphi_n)$ is not finite in general; for example, consider $X = [0, 1]$, $E = \mathbb{R}$ and the functions $\varphi_1 : [0, 1] \to \mathbb{R}$ and $\varphi_2 : [0,1] \to \mathbb{R}$ defined by putting, for each $x \in [0, 1]$, $\varphi_1(x) = x \operatorname{sen}(\pi/x)$ if $x \neq 0$, $\varphi_1(0) = 0$ and $\varphi_2(x) = 0$; then, for each $n \in \mathbb{N}$, $n \geqslant 1$, the function $\psi_n : [0,1] \to \mathbb{R}$ which takes the value 0 on the interval $\left[\dfrac{1}{n+1}, \dfrac{1}{n}\right]$ and agrees with φ_1 elsewhere, belongs to $A(\varphi_1, \varphi_2)$.

Moreover, if $\varphi_1, \ldots, \varphi_n$ have pairwise disjoint graphs but X has an infinite number of connected components, we have again that the set $A(\varphi_1, \ldots, \varphi_n)$ may be not finite $\left(\text{for example, consider } X = \{0\} \cup \bigcup_{n \geqslant 1} \left\{\dfrac{1}{n}\right\}\right)$; however, we have the following Proposition.

PROPOSITION 1.2. *Let* $\varphi_1, \ldots, \varphi_n$ $(n \geqslant 1)$ *be in* $\mathscr{C}(X, E)$ *with pairwise disjoint graphs.*

Then, the elements of $A(\varphi_1, \ldots, \varphi_n)$ *are all the functions in* $\mathscr{C}(X, E)$ *which coincide with some* $\varphi_i (i = 1, \ldots, n)$ *on every connected components of* X.

Consequently, if X *is a connected topological space, we have*

(1.5) $$A(\varphi_1, \ldots, \varphi_n) = \{\varphi_1, \ldots, \varphi_n\},$$

and, if X has m ($m \geq 1$) connected components, then

$$\operatorname{card}\,(A(\varphi_1,\ldots,\varphi_n)) = n^m.$$

Proof. Let C be a connected component of X and let $\varphi \in A(\varphi_1,\ldots,\varphi_n)$ and $x_0 \in C$; consider $i = 1,\ldots,n$ such that $\varphi(x_0) = \varphi_i(x_0)$ (cf. (1.4)) and the set $A = \{x \in C \mid \varphi(x) = \varphi_i(x)\}$. The set A is non empty and closed in the relative topology of C; moreover, if $x \in A$, there exists $V \in \mathfrak{B}$ such that, for each $j = 1,\ldots,n$, $j \neq i$, $\varphi_j(x) - \varphi_i(x) \notin V$; now, let $W \in \mathfrak{B}$ be such that $\overline{W} \subset V$ and $U \in \mathfrak{B}$ satisfying $U + U \subset W$. Since the functions $\varphi_j - \varphi_i$ are continuous ($j = 1,\ldots,n, j \neq i$), there exists a neighbourhood N of x such that $\varphi(y) - \varphi_i(y) \in U$ and $\varphi_j(y) - \varphi_i(y) \notin \overline{W}$ for each $y \in N$ and $j = 1,\ldots,n$, $j \neq i$. If, for some $y \in N$ and $j = 1,\ldots,n$, $j \neq i$, we have $\varphi_j(y) - \varphi(y) \in U$, it results $\varphi_j(y) - \varphi_i(y) = (\varphi_j(y) - \varphi(y)) + (\varphi(y) - \varphi_i(y)) \in U + U \subset W$ and this is a contradiction. Then, by (1.4), it must be $N \cap C \subset A$ and so A is open in the relative topology of C. Since C is connected, we obtain $A = C$, that is $\varphi = \varphi_i$.

Hence, (1.5) is true, and from a straighforward induction argument, we also obtain (1.6). ∎

In the sequel, we shall be concerned with continuous linear operators $L : \mathscr{C}(X, E) \to \mathscr{C}(X, E)$ satisfying the following property:

(1.7) *if $\varphi_1,\ldots,\varphi_n \in \mathscr{C}(X, E)$, there exist $\psi_1,\ldots,\psi_m \in A(\varphi_1,\ldots,\varphi_n)$ such that*

$$\varphi \in \mathscr{C}(X, E),\quad \varphi \in \operatorname{co}(\varphi_1,\ldots,\varphi_n)^* \Rightarrow L(\varphi) \in \operatorname{co}(L(\psi_1),\ldots,L(\psi_m)).$$

Obviously, the identity operator satisfies condition (1.7) (cf. [2, (1.1)]).

If we denote by Δ_m ($m \geq 1$) the set

(1.8) $$\Delta_m = \left\{(\lambda_1,\ldots,\lambda_m) \in \mathbb{R}^m \mid \lambda_i \geq 0 \text{ for each } i = 1,\ldots,m \text{ and } \sum_{i=1}^m \lambda_i = 1\right\},$$

condition (1.7) may be restated as follows

(1.9) *if $\varphi_1,\ldots,\varphi_n \in \mathscr{C}(X, E)$, there exist $\psi_1,\ldots,\psi_m \in A(\varphi_1,\ldots,\varphi_n)$ such that, for each $\varphi \in \mathscr{C}(X, E)$, $\varphi \in \operatorname{co}(\varphi_1,\ldots,\varphi_n)$ and $x_0 \in X$, there exists $(\lambda_1,\ldots,\lambda_m) \in \Delta_m$ such that $L(\varphi)(x_0) = L\left(\sum_{i=1}^m \lambda_i\psi_i\right)(x_0)$*, or equivalently

(1.10) *if $\varphi_1,\ldots,\varphi_n \in \mathscr{C}(X, E)$, there exist $\psi_1,\ldots,\psi_m \in A(\varphi_1,\ldots,\varphi_n)$ such that, for each $x_0 \in X$,*

$$\bigcup_{\varphi \in \mathscr{Sel}(\operatorname{co}(\varphi_1,\ldots,\varphi_n))} \{L(\varphi)(x_0)\} = \bigcup_{(\lambda_1,\ldots,\lambda_m)\in\Delta_m} \left\{L\left(\sum_{i=1}^m \lambda_i\psi_i\right)(x_0)\right\}.$$

Condition (1.10) says that, for each $f \in \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$, there exist $\psi_1,\ldots,\psi_m$ in $\mathscr{C}(X, E)$ such that $f = \operatorname{co}\,(\psi_1,\ldots,\psi_m)$ and, for each $x_0 \in X$,

$$\bigcup_{\varphi \in \mathscr{Sel}(f)} \{L(\varphi)(x_0)\} = \bigcup_{(\lambda_1,\ldots,\lambda_m)\in\Delta_m} \left\{L\left(\sum_{i=1}^m \lambda_i\psi_i\right)(x_0)\right\}.$$

* i.e. $\varphi(x) \in \operatorname{co}(\varphi_1(x),\ldots,\varphi_n(x))$ for each $x \in X$.

However, the functions $\psi_1,\ldots,\psi_m \in \mathscr{C}(X, E)$ such that $f = \operatorname{co}(\psi_1,\ldots,\psi_m)$ cannot be arbitrarily chosen. For example, consider the identity operator on $\mathscr{C}([-1,1], \mathbb{R})$ and the set-valued function $f : [-1,1] \to \mathscr{C}\mathscr{C}onv(\mathbb{R})$ defined by putting, for each $x \in [-1,1]$, $f(x) = [-x, x]$; then, the equality in (1.10) holds with $\psi_1(x) = -|x|$ and $\psi_2(x) = |x|$ ($x \in [-1,1]$), but not with $\psi_1(x) = -x$ and $\psi_2(x) = x$ ($x \in [-1,1]$).

If X is connected and $L : \mathscr{C}(X, E) \to \mathscr{C}(X, E)$ satisfies condition (1.7), by virtue of Proposition 1.2 and (1.7), we also have the following property

(1.11) *if $\varphi_1,\ldots,\varphi_n \in \mathscr{C}(X, E)$ have pairwise disjoint graphs, then*

$$\varphi \in \mathscr{C}(X, E),\quad \varphi \in \operatorname{co}\,(\varphi_1,\ldots,\varphi_n) \Rightarrow L(\varphi) \in \operatorname{co}\,(L(\varphi_1),\ldots,L(\varphi_n)).$$

and, by (1.10)

(1.12) *if $\varphi_1,\ldots,\varphi_n \in \mathscr{C}(X, E)$ have pairwise disjoint graphs, then, for each $x_0 \in X$,*

$$\bigcup_{\varphi \in \mathscr{Sel}(\operatorname{co}(\varphi_1,\ldots,\varphi_n))} \{L(\varphi)(x_0)\} = \bigcup_{(\lambda_1,\ldots,\lambda_n)\in\Delta_n} \left\{L\left(\sum_{i=1}^n \lambda_i\varphi_i\right)(x_0)\right\}$$

Condition (1.7) generalizes in abstract spaces the rôle of monotone operators in spaces of real valued continuous functions; in fact, we have the following Proposition.

Proposition 1.3. *If X is a connected compact Hausdorff topological space and $L : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ is a linear operator from $\mathscr{C}(X, \mathbb{R})$ in itself, the following statements are equivalent*

a) L satisfies condition (1.7);

b) there exist two closed subsets X^+ and X^- of X such that

$$X = X^+ \cup X^- \text{ and}$$

(1.13) $$\varphi \in \mathscr{C}(X, \mathbb{R}),\ \varphi \geq 0 \Rightarrow L(\varphi) \geq 0 \text{ on } X^+ \text{ and } L(\varphi) \leq 0 \text{ on } X^-.$$

Moreover, if L satisfies a) or equivalently b), we can take

(1.14) $$X^+ = L(1)^{-1}([0, \infty)),\quad X^- = L(1)^{-1}((-\infty, 0]),$$

where 1 denotes the constant function of constant value 1.

Proof. a) ⇒ b) Let X^+ and X^- be defined as in (1.14). Then X^+ and X^- are closed subsets of X and $X = X^+ \cup X^-$. Now, fix $\varphi \in \mathscr{C}(X, \mathbb{R})$, $\varphi \geq 0$; since L is linear, we can assume $\varphi \leq 1$ and therefore $\varphi \in \operatorname{co}(0, 1)$; by (1.11), we obtain $L(\varphi) \in \operatorname{co}\,(0, L(1))$, and consequently, for each $x \in X$, $L(\varphi)\,(x)$ must be positive if $L(1)\,(x)$ is positive (i.e. $x \in X^+$) and negative if $L(1)\,(x)$ is negative (i.e. $x \in X^-$).

b) ⇒ a) Let $\varphi_1,\ldots,\varphi_n \in \mathscr{C}(X, \mathbb{R})$, and consider $\psi_1 = \inf\,(\varphi_1,\ldots,\varphi_n)$ and $\psi_2 = \sup\,(\varphi_1,\ldots,\varphi_n)$. Then ψ_1 and ψ_2 are in $A(\varphi_1,\ldots,\varphi_n)$ (cf. 1.4)); if $\varphi \in \mathscr{C}(X, E)$, $\varphi \in \operatorname{co}\,(\varphi_1,\ldots,\varphi_n)$, we have $\psi_2 - \varphi \geq 0$ and $\varphi - \psi_1 \geq 0$; by (1.14), and the linearity of L, we obtain $L(\psi_1)\,(x) \leq L(\varphi)\,(x) \leq L(\psi_2)(x)$ if $x \in X^+$ and $L(\psi_2)\,(x) \leq L(\varphi)\,(x) \leq L(\psi_1)\,(x)$ if $x \in X^-$; in any case $L(\varphi) \in \operatorname{co}\,(L(\psi_1), L(\psi_2))$ and (1.7) is satisfied. ∎

Observe that condition (1.13) is equivalent to the following

(1.15) $\varphi, \psi \in \mathscr{C}(X, \mathbb{R}),\ \varphi \leqslant \psi \Rightarrow L(\varphi) \leqslant L(\psi)$ on X^+, $L(\varphi) \leqslant L(\psi)$ on X^-.

REMARK 1.4. 1. The implication b) $\Rightarrow$ a) in Proposition 1.3 remains true also if X is not connected, but the implication a) $\Rightarrow$ b) does not hold in general.

For example, consider $X = \{0, 1\}$ and the operator $L : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ defined by putting, for each $\varphi \in \mathscr{C}(X, \mathbb{R})$ and $x \in X$,

$$L(\varphi)\,(x) = \varphi(0) - \varphi(1). \tag{1.16}$$

Then, L does not satisfy condition b) in Proposition 1.3 since it takes opposite sign at the two positive functions χ_1 and χ_2 in $\mathscr{C}(X, \mathbb{R})$ defined by setting $\chi_1(0) = 0$, $\chi_1(1) = 1$ and $\chi_2(0) = 1$, $\chi_2(1) = 0$.

Now, we show that L satisfies condition a) in Proposition 1.3; let $\varphi_1, \ldots, \varphi_n \in \mathscr{C}(X, \mathbb{R})$ and consider the two functions ψ_1 and ψ_2 in $\mathscr{C}(X, \mathbb{R})$ defined by setting $\psi_1(0) = \inf\,(\varphi_1(0), \ldots, \varphi_n(0))$, $\psi_1(1) = \sup\,(\varphi_1(1), \ldots, \varphi_n(1))$ and $\psi_2(0) = \sup\,(\varphi_1(0), \ldots, \varphi_n(0))$, $\psi_2(1) = \inf\,(\varphi_1(1), \ldots, \varphi_n(1))$.

If $\varphi \in \mathscr{C}(X, \mathbb{R})$, $\varphi \in \mathrm{co}\,(\varphi_1, \ldots, \varphi_n)$, there exist $i_0, j_0, i_1, j_1 = 1, \ldots, n$ such that $\varphi_{i_0}(0) \leqslant \varphi(0) \leqslant \varphi_{j_0}(0)$, $\varphi_{i_1}(1) \leqslant \varphi(1) \leqslant \varphi_{j_1}(1)$ and hence $\psi_1(0) \leqslant \varphi(0) \leqslant \psi_2(0)$, $\psi_2(1) \leqslant \varphi(1) \leqslant \psi_1(1)$; by (1.16), we have $L(\psi_1) \leqslant L(\varphi) \leqslant L(\psi_2)$, that is $L(\varphi) \in \mathrm{co}(L(\psi_1), L(\psi_2))$.

2. By virtue of the preceding Proposition 1.3 and Remark 1.4.1, each monotone linear operator $L : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ satisfies condition (1.7).

In the case $E = \mathbb{R}$ observe that there exist continuous linear operators $L : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ which satisfy condition (1.7) and are not monotone; an example is furnished by the operator $L : \mathscr{C}([a, b], \mathbb{R}) \to \mathscr{C}([a, b], \mathbb{R})$ ($a, b \in \mathbb{R}$, $a < b$) defined by putting, for each $\varphi \in \mathscr{C}([a, b], \mathbb{R})$ and $x \in [a, b]$,

$$L(\varphi)\,(x) = \int_{x_0}^{x} \varphi(t)\,dt,$$

where x_0 is a fixed element in the open interval (a, b) (the proof that L satisfies (1.7) is based on the monotonicity of the integral and the fact that, for each $\varphi \in \mathscr{C}([a, b], \mathbb{R})$), the value that $L(\varphi)$ takes at $x \in [a, b]$ depends only on the values that φ takes in the interval with endpoints x and x_0).

However, condition (1.7) is not satisfied by every continuous linear operator $L : \mathscr{C}(X, E) \to \mathscr{C}(X, E)$; for example, consider again the real case $E = \mathbb{R}$ and the operator $L : \mathscr{C}([0, 1], \mathbb{R}) \to \mathscr{C}([0, 1], \mathbb{R})$ defined by putting for each $\varphi \in \mathscr{C}([0, 1], \mathbb{R})$ and $x \in [0, 1]$, $L(\varphi)(x) = \varphi(x_0) - \int_0^1 \varphi(t)dt$, where x_0 is fixed in $[0,1]$. Denote by φ_0 and φ_1 the constant functions of constant value 0 and respectively 1; then, for each $\lambda \in [0, 1]$ and $x \in [0, 1]$, $L(\lambda\varphi_0 + (1 - \lambda)\varphi_1))\,(x) = 0$ and (1.7) is not satisfied. ∎

In the next Proposition we shall show how it is possible to associate a continuous monotone linear operator $T_L : \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E)) \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$, from the subcone $\mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$ of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ in $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$, to every continuous linear operator $L : \mathscr{C}(X, E) \to \mathscr{C}(X, E)$ satisfying (1.7).

PROPOSITION 1.5. *Let $L : \mathscr{C}(X, E) \to \mathscr{C}(X, E)$ be a continuous linear operator satisfying* (1.7).

Then, for each $f \in \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$ and $x \in X$, the set

$$L_{f,x} = \bigcup_{\varphi \in \mathscr{S}el(f)} \{L(\varphi)\,(x)\} \tag{1.17}$$

is a non empty convex compact subset of E and the set-valued function $f_L : X \to \mathscr{C}\mathscr{C}onv(E)$ defined by putting, for each $x \in X$,

$$f_L(x) = L_{f,x} \tag{1.18}$$

is continuous.

Moreover, the map $T_L : \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E)) \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ defined by putting, for each $f \in \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$,

$$T_L(f) = f_L \tag{1.19}$$

is a continuous monotone linear operator from the subcone $\mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$ of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ in $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ satisfying conditions (2.2) *and* (2.3) *of* [2].

Proof. Let $f \in \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$ and consider $\psi_1, \ldots, \psi_m \in \mathscr{C}(X, E)$ as in (1.10). For each $x \in X$, the set $L_{f,x}$ is clearly non empty and convex; moreover, by (1.10), the set $L_{f,x}$ is the image of the continuous map $(\lambda_1, \ldots, \lambda_m) \mapsto L\left(\sum_{i=1}^{m} \lambda_i \psi_i\right)(x)$ defined on the compact set Δ_m (cf. (1.18)) and therefore $L_{f,x}$ is also compact.

Now, consider the set-valued function f_L defined as in (1.18). Let $x_0 \in X$ and $V \in \mathfrak{B}$; by the continuity of $L(\psi_1), \ldots, L(\psi_m)$ there exists a neighborhood N of x_0 such that, for each $x \in N$ and $i = 1, \ldots, m$,

$$L(\psi_i)(x) \in L(\psi_i)(x_0) + V,\quad L(\psi_i)(x_0) \in L(\psi_i)(x) + V;$$

then, for each $x \in N$ (cf. (1.10), (1.17) and (1.18)),

$$\begin{aligned} f_L(x) = L_{f,x} &= \bigcup_{\varphi \in \mathscr{S}el(f)} \{L(\varphi)\,(x)\} = \bigcup_{(\lambda_1, \ldots, \lambda_m) \in \Delta_m} \left\{ L\left(\sum_{i=1}^{m} \lambda_i \psi_i\right)(x) \right\} \\ &= \bigcup_{(\lambda_1, \ldots, \lambda_m) \in \Delta_m} \left\{ \sum_{i=1}^{m} \lambda_i L(\psi_i)(x) \right\} \subset \bigcup_{(\lambda_1, \ldots, \lambda_m) \in \Delta_m} \sum_{i=1}^{m} \lambda_i (L(\psi_i)(x_0) + V) \\ &\subset \left(\bigcup_{(\lambda_1, \ldots, \lambda_m) \in \Delta_m} \left\{ \sum_{i=1}^{m} \lambda_i L(\psi_i)(x_0) \right\} \right) + V \\ &= \bigcup_{(\lambda_1, \ldots, \lambda_m) \in \Delta_m} \left\{ L\left(\sum_{i=1}^{m} \lambda_i \psi_i\right)(x_0) \right\} + V = L_{f,x_0} + V = f_L(x_0) + V, \end{aligned}$$

and similarly,

$$f_L(x_0) \subset f_L(x) + V\,;$$

since x_0 is arbitrary in X, the set-valued function f_L is continuous (cf. [2, (1.6)]).

Now, consider the map T_L defined as in (1.19). The linearity of T_L follows from [2, Proposition 1.1] and Lemma 1.1; moreover, by (1.17), (1.18) and (1.19), it is clear that T_L is monotone and satisfies (2.2) and (2.3) of [2].

Thus, we have only to show that T_L is continuous. Let $V \in \mathfrak{V}$; since L is continuous there exists $U_1 \in \mathfrak{V}$ such that

(1.20) $\varphi \in \mathscr{C}(X, E)$, $\varphi(x) \in U_1$ for each $x \in X \Rightarrow L(\varphi)(x) \in V$ for each $x \in X$.

Now, let $U \in \mathfrak{V}$ be such that $\bar{U} \subset U_1$ and consider $f, g \in \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$ satisfying $f \leqslant g + U$, $g \leqslant f + U$. Let $x \in X$ and $y \in T_L(f)(x)$; by (1.17), (1.18) and (1.19) there exists a selection φ of f such that $y = L(\varphi)(x)$; since $f \leqslant g + U$, there exists a selection ψ of g such that $\varphi(t) - \psi(t) \in U_1$ for each $t \in X$ (indeed, it suffices to take a selection of the set-valued map $t \mapsto g(t) \cap (\varphi(t) - \bar{U})$ which is lower semicontinuous by virtue of [6, Proposition 2.5 and Proposition 2.3]); by (1.20), we have $L(\varphi)(t) - L(\psi)(t) \in V$ for each $t \in V$ and in particular, $L(\varphi)(x) - L(\psi)(x) \in V$; this yields $y = L(\varphi)(x) \in L(\psi)(x) + V \subset T_L(g)(x) + V$; since $y \in T_L(f)(x)$ and $x \in X$ are arbitrary, we obtain $T_L(f) \leqslant T_L(g) + V$; in a similar way, we have $T_L(g) \leqslant T_L(f) + V$ and this completes the proof. ■

At this point, we make the following Definition.

DEFINITION 1.6. *Let $L : \mathscr{C}(X, E) \to \mathscr{C}(X, E)$ be a continuous linear operator satisfying (1.7). We shall say that a subset Γ of $\mathscr{C}(X, E)$ is an $_0L$-Korovkin set in $\mathscr{C}(X, E)$ if, for each equicontinuous net $(L_\iota)^{\leqslant}_{\iota \in I}$ of linear perators from $\mathscr{C}(X, E)$ in itself satisfying condition (1.7) and such that the net $(L_\iota(\gamma))^{\leqslant}_{\iota\in I}$ converges to $L(\gamma)$ for each $\gamma \in \Gamma$, we also have that the net $(L_\iota(\varphi))^{\leqslant}_{\iota \in I}$ converges to $L(\varphi)$ for every $\varphi \in \mathscr{C}(X, E)$.*

If L is the identity operator, an L-Korovkin set in $\mathscr{C}(X, E)$ will be imply called a Korovkin set in $\mathscr{C}(X, E)$.

We are now in a position to state the following result.

THEOREM 1.7. *Let X be a connected compact Hausdorff topological space and $L : \mathscr{C}(X, E) \to \mathscr{C}(X, E)$ be a continuous linear operator satisfying condition (1.7).*

If a subset Γ of $\mathscr{C}(X, E)$ satisfies the following condition

(1.21) *for each $\varphi \in \mathscr{C}(X, E)$, $x_0 \in X$ and $V \in \mathfrak{V}$, there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ with pairwise disjoint graphs and such that*

$$\varphi \in \operatorname{co}(\gamma_1, \ldots, \gamma_n),$$

$$L(\gamma_1)(x_0), \ldots, L(\gamma_n)(x_0) \in L(\varphi)(x_0) + V,$$

then Γ is an L-Korovkin set in $\mathscr{C}(X, E)$.

Proof. Let $(L_\iota)^{\leqslant}_{\iota\in I}$ be an equicontinuous net of linear operators from $\mathscr{C}(X, E)$ in itself satisfying condition (1.7) and such that the net $(L_\iota(\gamma))^{\leqslant}_{\iota \in I}$

converges to $L(\gamma)$ for each $\gamma \in \Gamma$. Consider the operator $T \doteq T_L$ associated to L as in (1.19) and, for each $\iota \in I$, the operator $T_\iota = T_{L_\iota}$ associated to L_ι. Thus, we obtain the net $(T_\iota)^{\leqslant}_{\iota \in I}$ of continuous monotone linear operators from the subcone $\mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$ of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ in $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ and an operator T from $\mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E))$ in $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(E))$ satisfying conditions (2.2) and (2.3) of [2] (cf. Proposition 1.5).

Now, define the set

$H = \{h \in \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E)) \mid$ there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ with pairwise disjoint graphs and such that $h = \operatorname{co}(\gamma_1, \ldots, \gamma_n)\}$.

We show that the net $(T_\iota(h))^{\leqslant}_{\iota \in I}$ converges to $T(h)$ for each $h \in H$.

Let $h \in H$ and let $\gamma_1, \ldots, \gamma_n \in \Gamma$ with pairwise disjoint graphs such that $h = \operatorname{co}(\gamma_1, \ldots, \gamma_n)$.

Let $V \in \mathfrak{V}$; since the net $(L_\iota(\gamma_j))^{\leqslant}_{\iota \in I}$ converges to $L(\gamma_j)$ for each $j = 1, \ldots, n$, there exists $\alpha \in I$ such that, for each $j = 1, \ldots, n$, $x \in X$ and $\iota \in I$, $\iota \geqslant \alpha$,

$$L_\iota(\gamma_j)(x) \in L(\gamma_j)(x) + V, \quad L(\gamma_j)(x) \in L_\iota(\gamma_j)(x) + V.$$

By (1.12) and (1.19) we have, for each $x \in X$ and $\iota \in I$, $\iota \geqslant \alpha$,

$$T_\iota(h)(x) = \bigcup_{(\lambda_1, \ldots, \lambda_n) \in \Delta_n} \left\{ \sum_{j=1}^{n} \lambda_j L_\iota(\gamma_j)(x) \right\} \subset \bigcup_{(\lambda_1, \ldots, \lambda_n) \in \Delta_n} \sum_{j=1}^{n} \lambda_j (L(\gamma_j)(x) + V)$$

$$\subset \left(\bigcup_{(\lambda_1, \ldots, \lambda_n) \in \Delta_n} \left\{ \sum_{j=1}^{n} \lambda_j L(\gamma_j)(x) \right\} \right) + V = T(h)(x) + V$$

and similarly

$$T(h)(x) \subset T_\iota(h)(x) + V\,;$$

hence the net $(T_\iota(h))^{\leqslant}_{\iota \in I}$ converges to $T(h)$.

Taking into account that elements of $\mathfrak{V}$ are convex, we observe that condition (1.21) may be restated as follows

for each $\varphi \in \mathscr{C}(X, E)$, $x_0 \in X$ and $V \in \mathfrak{V}$, there exists $h \in X$ such that

$$\{\varphi\} \leqslant h, \quad T(h)(x_0) \subset T(\{\varphi\})(x_0) + V\,;$$

hence, we may argue as in the first part of the proof of [2, Theorem 2.4] to show that the net $(T_\iota(\{\varphi\}))^{\leqslant}_{\iota \in I}$ converges to $T(\{\varphi\})$ for each $\varphi \in \mathscr{C}(X, E)$ and this implies that the net $(L_\iota(\varphi))^{\leqslant}_{\iota \in I}$ converges to $L(\varphi)$ for each $\varphi \in \mathscr{C}(X, E)$. ■

REMARK 1.8. We point out that in (1.21) the requirement on $\gamma_1, \ldots, \gamma_n$ to have pairwise disjoint graphs is essential. For example, consider the operator $L : \mathscr{C}([a, b], \mathbb{R}) \to \mathscr{C}([a, b], \mathbb{R})$ ($a, b \in \mathbb{R}$, $a < b$) defined by putting, for each $\varphi \in \mathscr{C}([a, b], \mathbb{R})$ and $x \in [a, b]$, $L(\varphi)(x) = \int_a^b \varphi(t)\, dt$ and let

$$\Gamma = \left\{ \varphi \in \mathscr{C}([a, b], \mathbb{R}) \,\middle|\, \int_a^b \varphi(t)\, dt = 0 \right\}.$$

Then, for each $\varphi \in \mathscr{C}([a, b], \mathbb{R})$, $x_0 \in [a, b]$ and $\varepsilon > 0$, there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $\varphi(x) \in \mathrm{co}\ (\gamma_1(x), \gamma_2(x))$ and $\gamma_1(x_0) = \varphi(x_0) = \gamma_2(x_0)$. Moreover, L satisfies condition (1.7) (cf. Remark 1.4.1), but Γ is not a 0-Korovkin set in $\mathscr{C}([a, b], \mathbb{R})$ (indeed, $L \neq 0$).

REMARK 1.9. If X is a compact Hausdorff topological space with m ($m > 1$) connected components, and if we replace condition (1.21) with the following

(1.22) *for each* $\varphi \in \mathscr{C}(X, E)$, $x_0 \in X$ *and* $V \in \mathfrak{B}$, *there exist* $\gamma_1, \ldots, \gamma_m \in \mathscr{C}(X, E)$ *with pairwise disjoint graphs and such that*

$$A(\gamma_1, \ldots, \gamma_m) \subset \Gamma$$

$$\varphi \in \mathrm{co}\ (\gamma_1, \ldots, \gamma_m),$$

$$L(\gamma)\ (x_0) \in L\ (\varphi)\ (x_0) + V \textit{ for each } \gamma \in A(\gamma_1, \ldots, \gamma_m),$$

then Γ is an L-Korovkin set in $\mathscr{C}(X, E)$.

The proof is similar to that of Theorem 1.7, taking into account that the set $A(\gamma_1, .., \gamma_m)$ is finite for each $\gamma_1, .., \gamma_m \in \mathscr{C}(X, E)$ with pairwise disjoint graphs (cf. Propos ition 1.2) and therefore the net $(T_i(h))_{i \in I}^{\leq}$ converges to $T(h)$ for each $h \in H$, where H is defined as follows $H = \{h \in \mathscr{F}(X, \mathscr{C}\mathscr{C}onv(E)) \mid$ there exist $\gamma_1, \ldots, \gamma_m \in \mathscr{C}(X, E)$ with pairwise disjoint graphs such that $A(\gamma_1, \ldots, \gamma_m) \subset \Gamma$ and $h = \mathrm{co}(\gamma_1, \ldots, \gamma_m)\}$.

In the following Corollary we consider the special case where L is the identity operator.

COROLLARY 1.10. *If X is a connected compact Hausdorff topological space and a subset Γ of $\mathscr{C}(X, E)$ satisfies the following condition*
(1.23) *for each* $\varphi \in \mathscr{C}(X, E)$, $x_0 \in X$ *and* $V \in \mathfrak{B}$, *there exist* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *with pairwise disjoint graphs and such that*

$$\varphi(x) \in \mathrm{co}\ (\gamma_1, \ldots, \gamma_n)\ (x) \textit{ for each } x \in X,$$

$$\gamma_1(x_0), \ldots, \gamma_n(x_0) \in \varphi(x_0) + V,$$

then Γ is a Korovkin set in $\mathscr{C}(X, E)$.

At this point, we can obtain the well-known results in the case $E = \mathbb{R}$. We observe that if $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ we can define the functions $\alpha_f : X \to \mathbb{R}$ and $\beta_f : X \to \mathbb{R}$ by setting, for each $x \in X$,

$$\alpha_f(x) = \inf f(x), \qquad \beta_f(x) = \sup f(x);$$

due to the continuity of f, α_f and β_f are both continuous and $f = \mathrm{co}\ (\alpha_f, \beta_f)$; therefore the subcone $\mathscr{F}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ coincides with $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ (cf. (1.1)) and consequently, if we consider a monotone continuous linear operator $L : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$, the associated operator T_L (cf. Remark 1.4.1 and Proposition 1.6) is defined on the whole cone $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$. If X is connected, the proof of Theorem 1.7 also shows that the associated net $(T_{L_i}(f))_{i \in I}^{\leq}$ converges to $T_L(f)$ for each continuous set-valued function $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ (indeed, every continuous function $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ can be uniformly approximated by the continuous set-valued functions

$\mathrm{co}(\alpha_f - \varepsilon, \beta_f + \varepsilon)$ $(\varepsilon > 0)$, which satisfy $(\alpha_f - \varepsilon)\ (x) \neq (\beta_f + \varepsilon)(x)$ for each $x \in X$).

Moreover, we observe that if $L : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ is a monotone continuous linear operator and if $T_L : \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R})) \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ is the associated operator, by (1.17), (1.18) and (1.19) and Proposition 1.5, we obtain, for each $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ and $x \in X$,

$$T_L(f)\ (x) = [L(\alpha_f)(x),\ L(\beta_f)(x)]. \tag{1.24}$$

Taking into account the above remark, we can briefly return to consider set-valued continuous functions in order to give some examples of approximation processes in the cone $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$.

EXAMPLES 1.11. In the following examples we consider the set-valued Bernstein operators on the standard simplex and on the hypercube of $\mathbb{R}^p$ $(p \geq 1)$; on the standard simplex, the convergence of the sequence of these operators to the identity operator has been also obtained by Prolla [7] with different methods and quantitative estimates.

1. Consider $p \geq 1$ and let $X = X^p$ be the standard simplex in $\mathbb{R}^p$:

$$X^p = \left\{(x_1, \ldots, x_p) \in \mathbb{R}^p \mid x_i \geq 0 \text{ for each } i = 1, \ldots, p \text{ and } \sum_{i=1}^{p} x_i \leq 1\right\}.$$

For each $n \in \mathbb{N}$, we recall that the n-th Bernstein operator $B_n : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ is defined by setting, for each $\varphi \in \mathscr{C}(X, \mathbb{R})$ and $(x_1, \ldots, x_n) \in X$,

$$B_n(\varphi)\ (x_1, \ldots, x_p) = \sum_{\substack{h_1, \ldots, h_p \in \mathbb{N} \\ h_1 + \ldots + h_p \leq n}} \frac{n!}{h_1!, \ldots, h_p!(n - h_1 - \ldots - h_p)!} x_1^{h_1} \ldots x_p^{h_p} \left(1 - \sum_{i=1}^{p} x_i\right)^{n - h_1 - \ldots - h_p} \varphi\left(\frac{h_1}{n}, \ldots, \frac{h_p}{n}\right). \tag{1.25}$$

For each $n \in \mathbb{N}$, B_n is a positive continuous linear operator satisfying condition (1.7) and consequently we may consider the associated operator $B_n : \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R})) \to \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ which is a monotone continuous linear operator of $\mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ in itself. Since the sequence $(B_n(\varphi))_{n \in \mathbb{N}}$ converges to φ for each $\varphi \in \mathscr{C}(X, \mathbb{R})$, condition (1.21) in Theorem 1.7 is obviously satisfied with $\Gamma = \mathscr{C}(X, \mathbb{R})$ and consequently we can apply the same argument in the proof of Theorem 1.7 with $H = \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ to show that the sequence $(B_n(f))_{n \in \mathbb{N}}$ converges to f for each $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$.

In order to give an explicit expression of the set-valued Bernstein operators, we observe that, for each $n \in \mathbb{N}$, B_n is a positive operator and therefore we have, for each $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ and $(x_1, \ldots, x_p) \in X$ (cf. (1.24)),

$$B_n(f)\ (x_1, \ldots, x_p) = [B_n(\alpha_f)\ (x_1, \ldots, x_p),\ B_n(\beta_f)\ (x_1, \ldots, x_p)]; \tag{1.26}$$

we finally obtain, for each $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$ and $(x_1, \ldots, x_p) \in X$,

(1.27)
$$B_n(f)((x_1, \ldots, x_p)) = \sum_{\substack{h_1, \ldots, h_p \in \mathbb{N} \\ h_1 + \ldots + h_p \leqslant n}} \frac{n!}{h_1! \ldots h_p!(n - h_1 - \ldots - h_p)!} x_1^{h_1} \ldots x_p^{h_p} \left(1 - \sum_{i=1}^{p} x_i\right)^{n-h_1-\ldots-h_p} f\left(\frac{h_1}{n}, \ldots, \frac{h_p}{n}\right)$$

(the validity of (1.27) easily follows by denoting with $A_n(x_1, \ldots, x_p)$ the second member of (1.27) and, with the help of (1.25) and (1.26), by showing that the equivalence $y \in A_n(f)(x_1, \ldots, x_p) \Leftrightarrow y \in B_n(f)(x_1, \ldots, x_p)$ holds for an arbitrary $y \in \mathbb{R}$).

2. Consider $p \geqslant 1$ and let $X = [0,1]^p$ be the hypercube of $\mathbb{R}^p$. In this case, we recall that, for each $n \in \mathbb{N}$, the n-th Bernstein operator $B_n : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ is defined by setting, for each $\varphi \in \mathscr{C}(X, \mathbb{R})$ and $(x_1, \ldots, x_p) \in X$,

$$B_n(\varphi)(x_1, \ldots, x_p) = \sum_{h_1, \ldots, h_p = 0}^{n} \binom{n}{h_1} \cdots \binom{n}{h_p} x_1^{h_1}(1 - x_1)^{h_1} \ldots x_p^{h_p}(1 - x_p)^{h_p} \varphi\left(\frac{h_1}{n}, \ldots, \frac{h_p}{n}\right).$$

As in the first example, also in this case we have that the corresponding associated sequence $(B_n(f))_{n \in \mathbb{N}}$ converges to f for each $f \in \mathscr{C}(X, \mathscr{C}\mathscr{C}onv(\mathbb{R}))$.

Also in this case, the explicit expression of $B_n (n \in \mathbb{N})$ can be obtained following the same line of Example 1.11.1.

In the case $E = \mathbb{R}$, the classical definition of L-Korovkin set involves equicontinuous nets of monotone linear operators rather than linear operators satisfying condition (1.7); by virtue of Proposition 1.3 and Remark 1.4.1, and L-Korovkin set in the sense of Definition 1.6 is always an L-Korovkin in the classical sense.

By Proposition 1.3, Theorem 1.7 and Corollary 1.10, we obtain the following result which is well known in the case of monotone linear continuous operators (cf. [3] and [1, Theorem 3]).

Corollary 1.12. *Let X be a connected compact Hausdorff topological space and $L : \mathscr{C}(X, \mathbb{R}) \to \mathscr{C}(X, \mathbb{R})$ be a continuous linear operator of $\mathscr{C}(X, \mathbb{R})$ in itself satisfying the condition* b) *of Proposition* 1.3.

If a subset Γ of $\mathscr{C}(X, \mathbb{R})$ satisfies the following condition

(1.28) *for each $\varphi \in \mathscr{C}(X, \mathbb{R})$, $x_0 \in X$ and $\varepsilon > 0$, there exist γ_1, $\gamma_2 \in \Gamma$ such that $\gamma_1 \leqslant \varphi \leqslant \gamma_2$ and $|L(\gamma_1)(x_0) - L(\gamma_2)(x_0)| \leqslant \varepsilon$,*

then Γ is an L-Korovkin set in $\mathscr{C}(X, \mathbb{R})$.

Moreover, if a subset Γ of $\mathscr{C}(X, \mathbb{R})$ satisfies the following condition

(1.29) *for each $\varphi \in \mathscr{C}(X, \mathbb{R})$, $x_0 \in X$ and $\varepsilon > 0$ there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \leqslant \varphi \leqslant \gamma_2$ and $\gamma_2(x_0) - \gamma_1(x_0) \leqslant \varepsilon$,*

then Γ is a Korovkin set in $\mathscr{C}(X, \mathbb{R})$.

Proof. We have only to show that the functions γ_1 and γ_2 in (1.28) may be taken with disjoint graphs. Let $\varphi \in \mathscr{C}(X, \mathbb{R})$, $x_0 \in X$ and $\varepsilon > 0$ and consider $\delta > 0$ such that $\delta < \varepsilon/3$ and $\delta \mid L(1)(x) \mid \leqslant \varepsilon/3$ for each $x \in X$; define the functions $\varphi_1 = \varphi - \delta/2$ and $\varphi_2 = \varphi + \delta/2$; by (1.14), for each $i = 1, 2$, there exist $\gamma_{1,i}\ \gamma_{2,i} \in \Gamma$ such that $\gamma_{1,i} \leqslant \varphi_i \leqslant \gamma_{2,i}$ and $| L(\gamma_{1,i})(x_0) - L(\gamma_{2,i})(x_0) | \leqslant \delta$; then, the functions $\gamma_1 = \gamma_{1,1}$ and $\gamma_2 = \gamma_{2,2}$ have disjoint graphs and satisfy the conditions $\gamma_1 \leqslant \varphi \leqslant \gamma_2$ and

$$| L(\gamma_1)(x_0) - L(\gamma_2)(x_0) | \leqslant$$
$$\leqslant |L(\gamma_{1,1})(x_0) - L(\varphi_1)(x_0)| + |L(\gamma_{2,2})(x_0) - L(\varphi_2)(x_0)| +$$
$$+ |L(\varphi_1)(x_0) - L(\varphi_2)(x_0))| \leqslant$$
$$\leqslant | L(\gamma_{1,1})(x_0) - L(\gamma_{2,1})(x_0)| + | L(\gamma_{2,2})(x_0) - L(\gamma_{1,2})(x_0)| +$$
$$+ |L(\varphi_1)(x_0) - L(\varphi_2)(x_0)| \leqslant$$
$$\leqslant \delta + \delta + \varepsilon/3 \leqslant \varepsilon.$$

Finally (1.29) follows from (1.28) with L equal to the identity operator ■.

We can give another application of Theorem 1.7 and Corollary 1.10, by considering the particular case $E = \mathbb{R}^n$; for simplicity, we restrict our attention to the identity operator.

Corollary 1.13. *Let X be a connected compact Hausdorff topological space and Γ be a subset of $\mathscr{C}(X, \mathbb{R})$ satisfying condition* (1.29). *Then, the set*

(1.30) $\Gamma_n = \{\varphi \in \mathscr{C}(X, \mathbb{R}^n) \mid$ *there exists* $j = 1, \ldots, n$ *such that* $pr_j \circ \varphi \in \Gamma$ *and* $pr_i \circ \varphi = 0$ *for each* $i = 1, \ldots, n,\ i \neq j\}$

(where pr_i denotes the i-projection of $\mathbb{R}^n$ in $\mathbb{R}$) is a Korovkin set in $\mathscr{C}(X, \mathbb{R}^n)$.

Proof. Firstly, we observe that the functions γ_1 and γ_2 in (1.29) may be taken with disjoint graphs (cf. the proof of Corollary 1.12).

At this point, we denote by Λ_n the subspace generated by the subset Γ_n of $\mathscr{C}(X, \mathbb{R}^n)$ defined in (1.30) and we show that Λ_n satisfies condition (1.23); let $\varphi \in \mathscr{C}(X, \mathbb{R}^n)$, $x_0 \in X$ and $\varepsilon > 0$; for each $i = 1, \ldots, n$, by (1.29) there exist γ_i' and γ_i'' in $\mathscr{C}(X, \mathbb{R})$ with disjoint graphs and such that $\gamma_i' \leqslant \varphi_i \leqslant \gamma_i''$ ($\varphi_i = pr_i \circ \varphi$) and $\gamma_i''(x_0) - \gamma_i'(x_0) \leqslant \frac{1}{2} \varepsilon^{1/n}$. For each subset J of $\{1, \ldots, n\}$, consider the function $\gamma_J : X \to \mathbb{R}^n$ with components $\gamma_{J,i} (i = 1, \ldots, n)$ defined by

$$\gamma_{J,i} = \gamma_i' \text{ if } i \notin J \text{ and } \gamma_{J,i} = \gamma_i'' \text{ if } i \in J.$$

Then, for each $J \subset \{1, \ldots, n\}$, γ_J is the sum of n elements of Γ_n and therefore belongs to Λ_n; since γ_i' and γ_i'' have disjoint graphs for each $i = 1, \ldots, n$, the functions γ_J ($J \subset \{1, \ldots, n\}$) have pairwise disjoint graphs; moreover $\varphi \in co((\gamma_J)_{J \subset \{1, \ldots, n\}})$ and denoted by d the diameter of

$co((\gamma_J)_{J\subset\{1,\dots,n\}})(x_0)$, we have $d \leqslant \left(\frac{1}{2}\varepsilon^{1/n}\right)^n \leqslant \frac{1}{2}\varepsilon$, and hence $\gamma_J(x_0)$ ($J \subset \{1, \dots, n\}$) belongs to the closed ball in $\mathbb{R}^n$ of center $\varphi(x_0)$ and radius ε. Then Λ_n satisfies condition (1.23). ■

Finally, we observe that many examples of subsets Γ of $\mathscr{C}(X, \mathbb{R})$ satisfying condition (1.29) are well-known, and from them we can obtain many corresponding examples of subsets Γ_n of $\mathscr{C}(X, \mathbb{R}^n)$ defined as in (1.30) which are Korovkin sets in $\mathscr{C}(X, \mathbb{R}^n)$. Moreover, if we consider a subset Γ of $\mathscr{C}(X, \mathbb{R})$ satisfying (1.29) and consisting of p elements, the corresponding Korovkin set Γ_n in $\mathscr{C}(X, \mathbb{R}^n)$ consists exactly of pn elements; in particular, if X is the compact real interval $[0, 1]$, we can consider the minimum number $p = 3$ and obtain a Korovkin set in $\mathscr{C}(X, \mathbb{R}^n)$ consisting of $3n$ elements.

REFERENCES

1. Berens, H., and Lorentz, G. G., *Geometric theory of Korovkin sets*, J. Approx. Theory, **15** (1975), no. 3, 161–189.
2. Campiti, M., *Approximation of continuous set-valued functions in Fréchet spaces*, I, L'Analyse numérique et la théorie de l'approximation, **20** (1991), 1–2, 15–23.
3. Ferguson, L. B. O., and Rusk, M. D., *Korovkin sets for an operator on a space of continuous functions*, Pacific J. Math., **65** (1976), no. 2, 337–345.
4. Keimel, K., and Roth, W., *A Korovkin type approximation theorem for set-valued functions*, Proc. Amer. Math. Soc., **104** (1988), 819–823.
5. Keimel, K., and Roth, W., *Ordered cones and approximation*, preprint Technische Hochschule Darmstadt, part I, II, III, IV, 1988–89.
6. Michael, E., *Continuous selections*. I, Ann. Math., **63**, (1956), 2, 361–382.
7. Prolla, J. B., *Approximation of continuous convex-cone-valued functions by monotone operators*, preprint Universidade Estadual de Campinas, Brasil, no. 27 (1990).
8. Vitale, R. A., *Approximation of convex set-valued functions*, J. Approx. Theory, **26** (1979), no. 4, 301–316.

Received 1.IX.1990

Dipartimento di Matematica
Università degli Studi di Bari
Traversa 200 Via Re David, 4
70125 BARI (ITALY)