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A NEW REFINEMENT OF JENSEN'S INEQUALITY

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Abstract. In this paper we shall point out a new refinement of Jensen's discrete inequality. Certain applications in connection with some well-known results are also given.

In the recent paper [3], the first author established the following refinement of Jensen's inequality:

$$(1)\quad f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leqslant (\geqslant) \frac{1}{P_n^2}\sum_{i=1}^n\sum_{j=1}^n p_i p_j f\left(\frac{x_i+x_j}{2}\right) \leqslant (\geqslant) \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i),$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex (concave) mapping on I, x_i are in I and p_i are nonnegative numbers (I = an interval in $\mathbb{R}$, $i = 1, \ldots, n$) and $P_n := \sum_{i=1}^n p_i > 0$.

In this Note, we shall improve the "right" part of (1) as in the following. Certain applications are also given.

THEOREM. *Let $f : I \to \mathbb{R}$ and x_i, p_i $(i = 1, \ldots, n)$ be as above. Then one has the inequalities:*

$$(2)\quad \begin{aligned} \frac{1}{P_n^2}\sum_{i=1}^n\sum_{j=1}^n p_i p_j f\left(\frac{x_i+x_j}{2}\right) &\leqslant (\geqslant) \frac{1}{P_n^2}\sum_{i=1}^n\sum_{j=1}^n p_i p_j \int_0^1 f(tx_i + (1-t)x_j)\,dt \\ &\leqslant (\geqslant) \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \end{aligned}$$

Proof. Let $x_i \in I$, $p_i \geqslant 0$ so that $P_n > 0$ and consider the mapping $g : [0,1] \to \mathbb{R}$ given by

$$g(t) := \frac{1}{P_n^2}\sum_{i=1}^n\sum_{j=1}^n f(tx_i + (1-t)x_j).$$

Since f is convex on I, for all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ we have:

$$g(\lambda_1 t_1 + \lambda_2 t_2) = \frac{1}{P_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j f(x_j + (\lambda_1 t_1 + \lambda_2 t_2)(x_i - x_j)) =$$

$$= \frac{1}{P_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} f(\lambda_1 x_j + \lambda_1 t_1 (x_i - x_j) + \lambda_2 x_j + \lambda_2 t_2 (x_i - x_j)) \leq$$

$$\leq \frac{1}{P_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j [\lambda_1 f(x_j + t_1(x_i - x_j)) + \lambda_2 f(x_j + t_2(x_i - x_j))$$

$$= \lambda_1 g(t_1) + \lambda_2 g(t_2),$$

for all t_1, t_2 in $[0,1]$, i.e., g is also convex on $[0,1]$.

Applying Hadamard's inequalities for the convex mapping g (see e.g. [4]), we get:

$$\frac{1}{P_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right) = g\left(\frac{0+1}{2}\right) \leq \int_0^1 g(t)\, dt \leq$$

$$\leq \frac{g(0) + g(1)}{2} = \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i),$$

and the proof is finished.

The case of concave functions goes likewise and we omit it.

Now, we shall give some applications in connection to certain well-known inequalities in Mathematical Analysis.

Applications. 1. Let $x_i, p_i \geq 0$, $P_n > 0$, $i = 1, \ldots, n$. Then the following refinement of arithmetic-geometric inequality is valid:

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \geq \left[\prod_{i=1}^{n} \prod_{j=1}^{n} \left(\frac{x_i + x_j}{2}\right)^{p_i p_j}\right]^{1/P_n^2} \geq$$

$$\geq \exp \int_0^1 \ln \left[\prod_{i,j=1}^{n} (t x_i + (1-t) x_j)^{p_i p_j}\right]^{1/P_n^2} dt \geq \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}.$$

The proof is obvious from the above theorem for the convex mapping $f(x) = -\ln x$, $x > 0$.

2. Let $x_i \in \mathbb{R}$, $p_i \geq 0$ with $P_n > 0$ and $p \geq 1$. Then we have:

$$\left|\sum_{i=1}^{n} p_i x_i\right|^p \leq P_n^{p-2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \left|\frac{x_i + x_j}{2}\right|^p \leq$$

$$\leq P_n^{p-2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \int_0^1 |t x_i + (1-t) x_j|^p\, dt \leq P_n^{p-1} \sum_{i=1}^{n} p_i |x_i|^p.$$

The proof follows by the above theorem for the convex mapping $f: \mathbb{R} \to \mathbb{R}$, $f(x) := |x|^p$ $(p \geq 1)$.

3. The following refinement of the famous inequality due to Ky Fan (see [2, p. 5]) is also valid:

$$\sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1 - x_i) \geq \left[\prod_{i=1}^{n} \prod_{j=1}^{n} (x_i + x_j)/(2 - x_i - x_j)^{1/n^2} \geq\right.$$

$$\geq \exp \int_0^1 \ln \left\{\prod_{i,j=1}^{n} \left[\frac{(t x_i + (1-t) x_j)}{(1 - t x_i - (1-t) x_j)}\right]\right\}^{1/n^2} dt \geq \left[\prod_{i=1}^{n} x_i \Big/ \prod_{i=1}^{n} (1 - x_i)\right]^{1/n}$$

where $x_i \in (0, 1/2]$ and the equality holds in all inequalities iff $x_1 = \ldots \; x_n$.

The proof follows from the above theorem for the convex mapping $f: (0, 1/2] \to \mathbb{R}$, $f(x) = -\ln(x/(1-x))$.

4. In paper [1], H. Alzer has obtained the following converse of Ky Fan's inequality:

$$\sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1 - x_i) \leq \prod_{i=1}^{n} (x_i/(1 - x_i))^{x_i / \sum_{j=1}^{n} x_j}$$

where $x_i \in (0, 1)$, $i = 1, \ldots, n$. The equality holds in the above inequality iff $x_1 = \ldots = x_n$. By the use of the above theorem for the convex mapping $f(x) = \ln(x/(1-x))^x$, $x \in (0, 1)$, we have the refinement:

$$\sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1 - x_i) \leq \left\{\prod_{i,j=1}^{n} [(x_i + x_j)/(2 - x_i - x_j)]^{\frac{x_i + x_j}{2}}\right\}^{1 / \left(n \sum_{k=1}^{n} x_k\right)}$$

$$\leq \exp \int_0^1 \ln \left\{\prod_{i,j=1}^{n} (t x_i + (1-t) x_j)/(1 - t x_i - (1-t) x_j)]^{(t x_i + (1-t) x_j)}\right\}^{\frac{1}{n \sum_{k=1}^{n} x_k}} dt$$

$$\leq \prod_{i=1}^{n} (x_i/(1 - x_i))^{x_i / \sum_{k=1}^{n} x_k}$$

with equality in all inequalities iff $x_1 = \ldots = x_n$.

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