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APPROXIMATION OF ENTIRE HARMONIC FUNCTIONS IN R^3 HAVING INDEX-PAIR (p, q)

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Abstract. The authors have defined approximation error for harmonic functions in H_R, $0 < R < \infty$, the class of all harmonic functions H in R^3, that are regular in the open ball D_R of radius R centered at the origin and are continuous on the closure $\bar{D}_R$ of D_R; norm being the sup. norm. Necessary and sufficient conditions, in terms of the rate of decay of the approximation error $E_n(H, R)$, such that $H \in H_R$ has analytic continuation as an entire harmonic function having (p, q)-order ρ and lower (p, q)-order λ, have been obtained.

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0. Introduction. The harmonic functions in R^3 are the solutions of Laplace equation

$$(0.1) \qquad \frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \frac{\partial^2 H}{\partial x_3^2} = 0.$$

A harmonic function H, regular about the origin, can be expanded as

$$(0.2) \quad H \equiv H(r, \theta, \Phi) = \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} (a_{nm}^{(1)} \operatorname{Cos} m\Phi + a_{nm}^{(2)} \operatorname{Sin} m\Phi) P_n^m (\operatorname{Cos}\theta)$$

where $x_1 = r \operatorname{Cos} \theta$, $x_2 = r \operatorname{Sin} \theta \operatorname{Cos}\Phi$, $x_3 = r \operatorname{Sin}\theta \operatorname{Sin}\Phi$ and $P_n^m(t)$ are associated Legendre's functions of first kind of degree m and order n. A harmonic polynomial of degree K is a polynomial of degree k in x_1, x_2, and x_3 which satisfies (0.1).

A harmonic function H is said to be regular in

$D_R = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < R^2\}$, $0 < R \leqslant \infty$, if the series (0.2) converges uniformly on a compact subset of D_R. A harmonic function H is called entire if it is regular in D_∞.

The concepts of the index-pair (p,q), $p \geqslant q \geqslant 1$ and (p,q)-order, (p,q)-type etc. of an entire function were introduced by Juneja et al. ([4], [5]). Thus if we denote by $\log^{[p]} x$ the quantity $\log \log \ldots \log x$, where logarithm is taken p times, then an entire harmonic function H is said to be (p, q)-order ρ and lower (p,q)-order λ if it is of idex-pair (p, q) and

(0.3) $$\limsup_{r\to\infty}\ (\inf)\ \frac{\log^{[p]}M(r,H)}{\log^{[q]}r} = \frac{\rho(p, q)}{\lambda(p, q)} \equiv \frac{\rho(H)}{\lambda(H)};$$

$b \leqslant \lambda \leqslant \rho \leqslant \infty$. Here $b = 1$ if $(p,q) = (p,p)$, $p = 2, 3, \ldots$ and $b = 0$ otherwise, and $M(r, H) = \max\limits_{x_1^2+x_2^2+x_3^2=r^2} |H(x_1, x_2, x_3)|$.

The entire harmonic function H having (p,q)-order ρ, $b < \rho < \infty$, is said to be of (p,q)-type T if

(0.4) $$\limsup_{r\to\infty} \frac{\log^{[p-1]}M(r, H)}{(\log^{[q-1]}r)^{\rho}} = T(p, q) \equiv T(H),$$

where $0 \leqslant T \leqslant \infty$.

Fryant [2] related the order ρ and type T of an entire harmonic function H with the rate of decrease of coefficients $a_{nm}^{(i)}$ in (0.2), $i = 1,2$. Analogous results for the solutions of (0.1) which depend only on the variables $x = x_1$ and $y = (x_2^2 + x_3^2)^{1/2}$ have been found in Fryant [1] and Gilbert [3, Theorem, 4.3.4].

Let H_R, $0 < R < \infty$, denote the class of all harmonic function H regular in D_R and continuous on $\bar{D}_R$, the closure of D_R. For $H \in H_R$, let $E_n(H, R)$, the error in approximating the function H by harmonic polynomials of degree at most n in uniform norm, be defined as

(0.5) $$E_n(H, R) = \inf_{g\in\pi_n} || H - g ||_R$$

where π_n consist of all harmonic polynomials of degree at most n and

$$|| H - g ||_R = \max_{(x_1, x_2, x_3)\in \bar{D}_R} | H(x_1, x_2, x_3) - g(x_1, x_2, x_3) |.$$

Let $H \in H_R$. Kapoor and Nautiyal have obtained necessary and sufficient conditions for H to have an entire analytic continuation in terms of its order and type [6, Theorems 2 and 3]. These results obviously leave a big class of entire functions, such as entire functions of slow growth or of fast growth etc.

The aim of this paper is to extend [6, Theorem 2] for entire harmonic function of (p,q)-growth. We have also obtained analogous result for lower (p, q)-order of entire harmonic functions. Finally, we have studied the growth of polynomial expansion of entire harmonic functions with index-pair (p, q) in terms of approximation error.

We shall use the following notation throughout the paper.

NOTATION: $$P_\xi(\alpha) \equiv P_\xi(\alpha, p, q) = \begin{cases} \alpha & \text{if } p > q. \\ \xi + \alpha & \text{if } p = q = 2. \\ \max(1, \alpha) & \text{if } 3 \leqslant p = q < \infty. \\ \infty & \text{if } p = q = \infty. \end{cases}$$

where $0 \leqslant \alpha \leqslant \infty$ and $0 \leqslant \xi \leqslant 1$. We shall write $P(\alpha)$ for $P_1(\alpha)$.

1. Auxiliary results. In this section we give some lemmas that are used in proving Theorems 1,2.

LEMMA 1. *Associated Legendre's functions $P_n^m(t)$ satisfy*

(1.1) $$\max_{1\leqslant t\leqslant 1} | P_n^m(t) | \leqslant K[(n + m)!/(n - m)!]^{1/2},$$

where K is a constant independent of n and m.

LEMMA 2. *Let $H \in H_R$ be entire and $r' > 1$. Then, for all $r > 2r'R$ and all sufficiently large values of n, we have*

$$E_n(H,R) \leqslant \bar{K} M(r, H)\left(\frac{r'R}{r}\right)^{n+1}$$

Here $\bar{K}$ is a constant.

LEMMA 3. *Let $H \in H_R$. Then for any $R_* < R$ and $n \geqslant 1$, we have*

$$R_* \max_{m,\, i} \left[| a_{nm}^{(i)} | \left(\frac{(n + m)!}{(n - m)!}\right)^{1/2} \right] \leqslant K_0(2n + 1)E_{n-1}(H, R)$$

where K_0 is a constant.

The proof of above lemmas can be found in [6, pp. 1026 – 27].

LEMMA 4. *Let $H \in H_R$. Then for any $R_* < R$ and $n \geqslant 1$, there exists an entire function $h(z)$ such that*

$$h(z) = \sum_{n=1}^{\infty} (2n + 1)^2 E_{n-1}(H, R)\left(\frac{Z}{R_*}\right)^n,$$

and

$$M(r, H) \leqslant | a_{00}^{(1)} | + KK_0M(r, h),$$

where $M(r, h) = \max\limits_{|z|\leqslant r} | h(z) |$.

Proof. For $H \in H_R$, using (0.2), Lemma 1 and Lemma 3, we have

$$\left| \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} (a_{nm}^{(1)} \operatorname{Cos} m\,\Phi + a_{nm}^{(2)} \operatorname{Sin} m\,\Phi)P_n^m (\operatorname{Cos} \theta) \right|$$

$$\leqslant |a_{00}^{(1)}| + K \sum_{n=0}^{\infty} (2n+1)\, r^n \max_{m,\,i} \Big]\, |a_{nm}^{(i)}| \left(\frac{(n+m)!}{(n-m)!} \right)^{1/2} \Big]$$

$$\leqslant |a_{00}^{(1)}| + KK_0 \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H,\ R) \left(\frac{r}{R_*} \right)^n ,$$

for some $R_* < R$,

$$M(r,\ H) \leqslant |a_{00}^{(1)}| + KK_0 M(r,\ h).$$

where

$$h(z) = \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R) \left(\frac{Z}{R_*} \right)^n ,$$

Since $\lim\limits_{n\to\infty} (E_n(H,\ R))^{1/n} = 0$, $h(z)$ is an entire function of a single complex variable z and $M(r,\ h) = \max\limits_{|z|\leqslant r} |h(z)|$.

2. Main results

THEOREM 1. *Let $H \in H_R$. Then, H has analytic continuation as an entire harmonic function of finite $(p,\ q)$-order ρ such that*

$$\rho(p,\ q) = P(L(p,\ q))$$

where

$$L(p,\ q) = \lim_{n\to\infty} \sup \frac{\log^{[p-1]} n}{\log^{[q]}(E_n(H,\ R))^{-1/n}}$$

Proof. First assume that $\rho(p,\ q) < \infty$. Then for any $\in > 0$

$$\log M(r,\ H) < \exp^{[p-2]}(\log^{[q-1]} r)^{\rho+\in} \tag{2.1}$$

for all $r > r_0 = r_0(\in)$. Using Lemma 3, we have

$$E_n(H,\ R) \leqslant \bar{K} M\,(r,\ H) \left(\frac{r'R}{r} \right)^{n+1} ,$$

or

$$\log M(r,\ H) \geqslant \log E_n(H,\ R) + (n+1) \log \left(\frac{r}{r'R} \right) - \log \bar{K} ,$$

or

$$\exp^{[p-2]}(\log^{[q-1]} r)^{\rho+\in} > \log E_n(H,\ R) + (n+1) \log \left(\frac{r}{r'R} \right) - \log \bar{K} .$$

$$\log E_n(H,\ R) < \exp^{[p-2]}\, (\log^{[q-1]} r)^{\rho+\in} - (n+1) \log \left(\frac{r}{r'R} \right) + \log \bar{K} . \tag{2.2}$$

For $(p,q) = (2,1)$ we proceed on the lines of Kapoor and Nautiyal [6] to get

$$\rho(2,1) \geqslant L(2,1). \tag{2.3}$$

For $(p,q) = (2,2)$, from (2.2) we have

$$\log E_n(H,\ R) < (\log r)^{\rho+\in} - (n+1) \log \left(\frac{r}{r'R} \right) + \log \bar{K}.$$

Choosing

$$\frac{r}{r'R} = \exp \left(\frac{n}{\rho+\in} \right)^{1/\rho-1+\in}$$

in the above inequality, we get

$$\log E_n(H,\ R) < \left[\left(\frac{n}{\rho+\in} \right)^{1/\rho-1+\in} + \log r'R \right]^{\rho+\in} -$$

$$-(n+1) \left(\frac{n}{\rho+\in} \right)^{1/\rho-1+\in} + \log \bar{K}$$

$$= \left(\frac{n}{\rho+\in} \right)^{\rho+\in/(\rho-1+\in)} \left[[1+0(1)]^{\rho+\in} - (\rho+\in) - \frac{\rho+\in}{n} \right]$$

$$+ \log \bar{K} - \frac{1}{n} \log E_n(H, R)$$

$$> \left(\frac{n}{\rho+\in} \right)^{1/\rho-1+\in} \left\{ 1 + \frac{1}{n} - \frac{1}{\rho+\in} [1+0(1)]^{\rho+\in} \right\} - \frac{1}{n} \log \bar{K}$$

or

$$\log \log (E_n(H,\ R))^{-1/n} > \frac{1}{\rho-1+\in} \log n + 0(1),$$

or

$$\frac{\log n}{\log \log (E_n(H,\ R))^{-1/n}} < (\rho-1+\in) \frac{1}{[1+0(1)]} \text{ as } n\to\infty,$$

Proceeding to limits we get

$$\rho(2,2) \geqslant 1 + L(2,2). \tag{2.4}$$

Now for $(p,q) \neq (2,1)$ and $(2,2)$, let

$$\frac{r}{r'R} = \exp^{[q-1]} \left[\log^{[p-2]} \left(\frac{n}{\rho+\in} \right) \right]^{1/\rho+\in} .$$

Using (2.2) we get

$$\log E_n(H, R) < \frac{n}{\rho+\epsilon} - (n+1)\exp^{[q-2]}\left[\log^{[p-2]}\left(\frac{n}{\rho+\epsilon}\right)\right]^{1/\rho+\epsilon}$$

$$-\frac{1}{n}\log E_n(H, R) > \exp^{[q-2]}\left(\log^{[p-2]}\left(\frac{n}{\rho+\epsilon}\right)\right)^{1/\rho+\epsilon}(1+0(1)) - \frac{1}{\rho+\epsilon},$$

or

$$\log^{[q-1]}(\log(E_n(H, R))^{-1/n}) > \frac{1}{\rho+\epsilon}\log^{[p-1]}\frac{n}{\rho+\epsilon},$$

or

$$\frac{\log^{[p-1]}(n/\rho+\epsilon)}{\log^{[q]}(E_n(H, R))^{-1/n}} < \rho+\epsilon+0(1) \text{ as } n\to\infty.$$

Proceeding to limits we get

(2.5) $$\rho(p,q) \geqslant L(p, q).$$

Since $\rho(p,q) \geqslant 1$ for $p = q$, the above inequality [2.2] for $3 \leqslant p = q < \infty$ gives

(2.6) $$\rho(p, p) \geqslant \max(1, L(p, p)).$$

and for $p > q$ it gives

(2.7) $$\rho(p, q) \geqslant L(p, q).$$

Combining (2.4), (2.5) and (2.6) we obtain

(2.8) $$\rho(p,q) \geqslant P(L(p, q)).$$

(2.8) holds obviously if $\rho(p, q) = \infty$.
Next assume that $L(p,q) < \infty$. From any $\epsilon > 0$,

(2.9) $$E_n(H, R) < \exp\{-n\exp^{[q-2]}(\log^{[p-2]}n)^{1/L+\epsilon}\}.$$

By Lemma 3, we have

$$M(r, H) \leqslant |a_{00}^{(1)}| + KK_0\sum_{n=1}^{\infty}(2n+1)^2E_{n-1}(H, R)\left(\frac{r}{R_*}\right)^n \text{ for } R > R_*$$

(2.10) $$M(r, H) \leqslant A_0 + B(n_0) + KK_0\sum_{n=N+1}^{n(r)}(2n+1)^2E_{n-1}(H, R)\left(\frac{r}{R_*}\right)^n$$

$$+ KK_0\sum_{n=n(r)+1}^{\infty}(2n+1)^2E_{n-1}(H, R)\left(\frac{r}{R_*}\right)^n,$$

where $B(n_0)$ is a polynomial of degree at most n.
Choose

$$\exp\{-n\exp^{[q-2]}(\log^{[p-2]}n)^{1/L+\epsilon}\}\left(\frac{r}{R^*}\right)^n < \frac{1}{2^n} \text{ for } n \geqslant n(r)$$

and

$$n(r) = \exp^{[p-2]}\left\{\left(\log^{[q-1]}\frac{2r}{R^*}\right)^{L+\epsilon}\right\}.$$

Using (2.9) and (2.10) we have

$$M(r, H) < A_0 + B(n_0) + KK_0\left(\frac{r}{R^*}\right)\exp^{[p-2]}\left\{\left[\log^{[q-1]}\left(\frac{2r}{R^*}\right)\right]^{L+\epsilon}\right\}$$

$$\sum_{k=0}^{\infty}(2k+1)^2\exp\{-k\exp^{[q-2]}[\log^{[p-2]}n]^{1/L+\epsilon}\}$$

$$+ KK_0\sum_{k=0}^{\infty}(2k+1)^2\frac{1}{2^k}.$$

Since both the series in the Right hand expression are convergent, we have for large value of r,

(2.11) $$\log^{[2]}M(r, H) < \exp^{[p-3]}\left\{\log^{[q-1]}\left(\frac{2r}{R^*}\right)^{L+\epsilon}\right\} + \log\log r + 0(1)$$

For $(p,q) = (2,1)$

$$\log^{[2]}M(r, H) < (L+\epsilon)\log\left(\frac{2r}{R^*}\right) + \log\log r + 0(1),$$

or

$$\frac{\log^{[2]}M(r, H)}{\log r} < (L+\epsilon)\frac{\log\left(\frac{2r}{R^*}\right)}{\log r} + \frac{\log\log r}{\log r} + 0(1).$$

Proceeding to limits we get

(2.12) $$\rho(2,1) \leqslant L(2, 1).$$

For $(p, q) = (2,2)$, we have

$$\frac{\log^{[2]}M(r, H)}{\log^{[2]}r} < (L+\epsilon)\frac{\log^{[2]}\left(\frac{2r}{R^*}\right)}{\log^{[2]}r} + 1 + 0(1).$$

Proceeding to limits we get

$$\rho(2,2) \leqslant L(2,2) + 1. \tag{2.13}$$

Since $\rho \geqslant 1$ for $p = q$ the inequality (2.11) for $p = q \geqslant 3$ gives

$$\rho(p, p) \leqslant \max\ (1,\ L(p, p)).$$

and for $p > q$ it gives

$$\rho(p, q) \leqslant L(p, q).$$

therefore

$$\rho(p, q) \leqslant P(L(p, q)). \tag{2.14}$$

This inequality is obviously true if $L(p, q) = \infty$. Combining (2.8) and (2.14) we get $\rho(p, q) = P(L(p, q))$. This completes the proof of Theorem 1.

THEOREM 2. *Let $H \in H_R$. Then H has analytic continuation as an entire harmonic function of finite lower (p, q)-order λ, such that*

$$\lambda(p,q) = P(l(p, q))$$

where

$$l(p, q) = \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]}(E_n(H, R))^{-1/n}}. \tag{2.15}$$

Proof. We note that $0 \leqslant l \leqslant \infty$. First let $0 < l < \infty$. Then for $l > \epsilon > 0$ and for all $n > n_0 = n_0(\epsilon)$,

$$E_n(H, R) > \exp\{-n \exp^{[q-2]}(\log^{[p-2]} n)^{1/l-\epsilon}\}. \tag{2.16}$$

Using Lemma 3, and (2.16) we get

$$\bar{K} M(r, H) \left(\frac{r'R}{r}\right)^{n+1} > \exp\{- n \exp^{[q-2]}(\log^{[p-2]} n)^{1/l-\epsilon}\}$$

or

$$\log M(r, H) > - n \exp^{[q-2]}(\log^{[p-2]} n)^{1/l-\epsilon} + (n + 1) \log\left(\frac{r}{r'R}\right) - \log \bar{K}. \tag{2.17}$$

For $(p, q) = (2,1)$

$$\log M(r, H) > - n \log (n)^{1/l-\epsilon} + (n + 1) \log\left(\frac{r}{r'R}\right) - \log \bar{K}.$$

Choosing

$$\frac{r}{r'R} = 2(n)^{1/l-\epsilon}$$

in above inequality, we get

$$\log M(r, H) > (n + 1) \log 2 + \log (n)^{1/l-\epsilon} - \log \bar{K}.$$

$$= \left(\frac{r}{2r'R}\right)^{l-\epsilon} \log 2 + \log\left(\frac{r}{r'R}\right) - \log \bar{K},$$

$$\log\log M(r, H) > (l - \epsilon) \log\left(\frac{r}{2r'R}\right) [1 + 0(1)] \qquad , \text{as } r \to \infty$$

or

$$\frac{\log\log M(r, H)}{\log r} > (l - \epsilon) \frac{\log\left(\dfrac{r}{2r'R}\right)}{\log r} [1 + 0(1)] \qquad , \text{as } r \to \infty$$

Proceeding to limits we get

$$\lambda(2,1) \geqslant l(2,1). \tag{2.18}$$

For $(p,q) = (2,2)$ we observe that $1 \leqslant l < \infty$ and choose

$$\frac{r}{r'R} = \exp[2(n)^{1/l-\epsilon}].$$

Proceeding as above, from (2.17) we get

$$\log M(r, H) > n^{\frac{l-\epsilon+1}{l-\epsilon}} + 2n^{1/l-\epsilon} - \log \bar{K}$$

$$= \frac{1}{2^{l-\epsilon+1}} \left(\log\left(\frac{r}{r'R}\right)\right)^{l-\epsilon+1} + \log\left(\frac{r}{r'R}\right) - \log \bar{K}$$

which gives

$$\frac{\log\log M(r, H)}{\log\log r} > (l - \epsilon + 1) \frac{\log\log (r/r'R)}{\log\log r} + 0(1), \qquad \text{as } r \to \infty.$$

Proceeding to limits we get

$$\lambda(2.2) \geqslant l(2,2) + 1 \tag{2.19}$$

For $p = q \geqslant 3$,

Choosing

$$\frac{r}{r'R} = 2 \exp^{[q-1]}(\log^{[p-2]} n)^{1/l-\in}.$$

By (2.17) we have

$$\log M(r, H) > \exp^{[q-2]}(\log^{[p-2]} n)^{1/l-\in} + (n+1) \log 2 - \log \bar{K}.$$

$$= \log\left(\frac{r}{r'R}\right) + \{\exp^{[p-2]}(\log^{[q-1]} r/2)\}^{l-\in} \log 2 - \log \bar{K}.$$

Therefore, for sufficiently large value of r,

$$\log^{[p]} M(r, H) > (l - \in) \log^{[q]} r/2 + 0(1).$$

Since $\lambda \geqslant 1$ for $p = q$, this inequality gives, for $p = q \geqslant 3$

$$\text{(2.20)} \qquad \liminf_{r\to\infty} \frac{\log^{[p]} M(r, H)}{\log^{[p]} r} \geqslant \max(1, l(p, p))$$

and $p > q$

$$\liminf_{r\to\infty} \frac{\log^{[p]} M(r, H)}{\log^{[q]} r} \geqslant l(p, q).$$

(2.18), (2.19) and (2.20) together prove $\lambda \geqslant P(l(p,q))$. It is obviously true if $l = 0$. If $l = \infty$, the above arguments can be repeated with an arbitrary large number in place of $(l - \in)$ to give $\lambda = \infty$.

Next to prove reverse inequality, for $H \in H_R$, using Lemma 4, we get

$$M(r, H) \leqslant |a_{00}^{(1)}| + KK_0 \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R)\left(\frac{r}{R_*}\right)^n,$$

$$\text{for } R > R_*.$$

$$= |a_{00}^{(1)}| + KK_0 M(r, h),$$

where

$h(z) = \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R)\left(\frac{Z}{R_*}\right)^n$, since $h(z)$ is an entire function of a single complex variable z and $M(r, h) = \max_{|z|\leqslant r} |h(z)|$, applying the formula expressing the lower (p,q)-order λ of an entire function of a single complex variable in terms of its Taylor coefficients to the function $h(z)$ we obtain,

$$\lambda(p, q) \equiv \text{lower } (p,q)\text{-order of } H \leqslant \text{lower } (p, q)\text{-order of } h(z) \equiv \lambda_1(p, q) \text{ say.}$$

It follows that

$$\text{(2.21)} \qquad \lambda(p,q) \leqslant P(l(p, q)).$$

This completes proof of Theorem 2.

REFERENCES

1. Fryant, A. J., *Growth and complete sequences of generalized axisymmetric potentials.* J. Approx. Theory, **19** (1977), 361–370.
2. Fryant, A. J., *Growth of entire harmonic functions in* $\mathbf{R}^3$. J. Math. Anal. Appl., **66** (1978), 599–605.
3. Gilbert, R. P., *Function theoretic methods in partial differential equations.* Academic Press, New York, 1969.
4. Juneja, O. P., Kapoor G. P. and Bajpai, S. K., *On the (p, q)-order and lower (p, q)-order of an entire function.* J. reine. angew. Math., **282** (1976), 53–57.
5. Juneja, O. P., Kapoor G. P. and Bajpai, S. K., *On the (p, q)-type and lower (p, q)-type of an entire function.* J. reine angew. Math., **290** (1977), 180–190.
6. Kapoor, G. P. and Nautiyal, A., *Approximation of entire harmonic functions in* $\mathbf{R}^3$. Indian J. pure appl. Math., **13** (1982), 1024–1030.

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