

INTER-RELATIONS AMONG MULTIFUNCTIONS DEFINED ON BITOPOLOGICAL SPACES

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Abstract. This paper establishes links between weakly continuous, quasicontinuous and H-almost continuous multifunctions defined on bitopological spaces with values on bitopological spaces, generalizing the results from [3], [6], [8], [12], [13] and [15].

1. Introduction

The notions of upper (lower) weakly continuous and H-upper (lower) almost continuous multifunctions were introduced in 1978 in [11] and [17]. The notion of weakly continuous function from [5] was extended in 1982 to the function defined on bitopological spaces in [2] and for multifunctions defined on bitopological spaces in [14].

In [15] are established the links between quasicontinuous, H-almost continuous and weakly continuous multifunctions between topological spaces.

There are recent researches about quasicontinuous and H-almost continuous multifunctions defined on topological spaces with values on bitopological spaces in [3], and defined on bitopological spaces with values on topological spaces in [6].

In this paper we will establish links between weak continuous, quasicontinuous and H-almost continuous defines on bitopological spaces with values in bitopological spaces, generalizing the results from [3], [6] and the results from [8], [12], [13] and [15].

2. Definition. Notation

Definition 1 [1]. Let (X, P, Q) be a bitopological space. We say that $A \subset X$ is (1, 2)-semi open if there exists a P -open set B such that $B \subset A \subset Cl_Q(B)$.

Similarly A is (2, 1)-semi open if there exists a Q -open set B such that $B \subset A \subset Cl_P(B)$.

Definition 2 [1]. Let (X, P, Q) be a bitopological space. We say that $A \subset X$ is (1, 2)-semi closed if the set $(X-A)$ is (1, 2)-semi open. Similarly A is (2, 1)-semi closed if $(X-A)$ is (2, 1)-semi open.

Definition 3 [4]. Let (X, P, Q) be a bitopological space. Then P is regular with respect to Q if for each point $x \in X$ and each P -closed set A such that $x \notin A$ there is a P -open set U and a Q -open set V disjoint

from U such that $x \in U$ and $P \subset V$. (X, P, Q) is pairwise regular if P is regular with respect to Q and Q is regular with respect to P .

Definition 4 [8]. A set A of a bitopological space (X, P, Q) is called strictly (P, Q) paracompact if every cover ν of A with P -open sets has a refinement ν' with P -open sets, which cover A and ν' is Q locally finite.

Similarly we define A to be strictly (Q, P) paracompact.

Definition 5. A multifunction $F: (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is said to be:

a) $(2,1) - H$ upper almost continuous ($(2,1) - H$. u.a.c.) at $x_0 \in X_1$ if for each Q_2 -open set G containing $F(x_0)$,

$$x_0 \in \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^+(G))).$$

b) $(2,1) - H$ lower almost continuous ($(2,1) - H$. l. a. c.) at $x_0 \in X_1$ if for each $Q_2 -$ open set G such that $F(x_0) \cap G \neq \emptyset$,

$$x_0 \in \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^-(G))).$$

c) $(2,1) - H$. u. a. c. ($(2,1) - H$. l. a. c.) if it has this property in any point $x \in X_1$.

If F is $(2,1)$ and $(1,2) - H$. u. a. c. (H . l. a. c.) then F is pairwise H . u. a. c. (pairwise H . l. a. c.).

Definition 6. A multifunction $F: (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is said to be:

a) $(1,2) -$ upper quasicontinuous ($(1,2) -$ u. q. c.) at $x_0 \in X_1$ if for each $P_2 -$ open set V containing $F(x_0)$ there exists a $(1,2) -$ semi open set U containing x_0 such that $F(U) \subset V$.

b) $(1,2) -$ lower quasicontinuous ($(1,2) -$ l. q. c.) at $x_0 \in X_1$ if for each $P_2 -$ open set V such that $F(x_0) \cap V \neq \emptyset$ there exists a $(1,2) -$ semi open set U containing x_0 such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

c) $(1,2) -$ u. q. c. ($(1,2) -$ l. q. c.) if it has this property in any point $x \in X_1$.

If F is $(1,2)$ and $(2,1) -$ u. q. c. (l . q. c.), then F is said pairwise u. q. c. (pairwise l . q. c.).

Definition 7. [14]. A multifunction $F: (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is said to be:

a) $(1,2) -$ upper weakly continuous ($(1,2) -$ u. w. c.) at $x_0 \in X_1$ if for each $P_2 -$ open set G containing $F(x_0)$ there exists a $P_1 -$ open set U containing x_0 such that $F(U) \subset \text{Cl}_{Q_2}(G)$.

b) $(1,2) -$ lower weakly continuous ($(1,2) -$ l. w. c.) at $x_0 \in X_1$ if for each P_2 open set G such that $F(x_0) \cap G \neq \emptyset$ there exists a $P_1 -$ open set U containing x_0 such that $F(u) \cap \text{Cl}_{Q_2}(G) \neq \emptyset$ for every $u \in U$.

c) $(1,2) -$ u. w. c. ($(1,2) -$ l. w. c.) if it has this property in any point $x \in X_1$.

If F is $(1,2)$ and $(2,1) -$ u. w. c. (l . w. c.) then F is said pairwise u. w. c. (pairwise l . w. c.).

3. Properties

Lemma 1. The following are equivalent for a multifunction $F: (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$:

1. F is $(1,2) -$ u. q. c.

2. $F^+(G)$ is $(1,2) -$ semi open for every $P_2 -$ open set $G \subset X_2$.

3. $F^-(V)$ is $(1,2) -$ semi closed for every $P_2 -$ closed set $V \subset X_2$.

Proof. (1) \Rightarrow (2). Let $x \in X_1$ and G be any $P_2 -$ open set in X_2 such that $F(x_0) \subset G$. Then there is a $(1,2) -$ semi open set U_x containing x and $F(U_x) \subset G$. Then $F^+(G) = \bigcup \{U_x : x \in F^+(G)\}$. By [1, Theorem 1.2] $F^+(G)$ is $(1,2) -$ semi open.

(2) \Rightarrow (1). Let $x \in X_1$ and G be any $P_2 -$ open set such that $F(x_0) \subset G$. Then $x \in F^+(G)$ and $F^+(G)$ is $(1,2) -$ semi open. Let $U = F^+(G)$ be, then $x \in U$, U is $(1,2) -$ semi open and $F(U) \subset G$. So F is $(1,2)$ u. q. c. at x .

(2) \Leftrightarrow (3). Follows from $F^+(X_2 - G) = X_1 - F^-(G)$.

Lemma 2. The following are equivalent for a multifunction $F: (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$:

1. F is $(1,2) -$ l. q. c.

2. $F^-(G)$ is $(1,2) -$ semi open for each $P_2 -$ open set $G \subset X_2$.

3. $F^+(V)$ is $(1,2) -$ semi closed for every $P_2 -$ closed set $V \subset X_2$.

Proof. It is similar to the proof of Lemma 1.

Theorem 1. If the multifunction $F: (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is $(1,2) -$ u. q. c. and $(2,1) - H$. l. a. c., then F is $(2,1) -$ l. w. c.

Proof. Let $x \in X_1$ and G be any $Q_2 -$ open set such that $F(x) \cap G \neq \emptyset$. Then $X_2 - \text{Cl}_{P_2}(G)$ is a $P_2 -$ open set and by Lemma 1, $F^+(X_2 - \text{Cl}_{P_2}(G))$ is $(1,2) -$ semi open. By [1, Theorem 1.1] we have:

$$F^+(X_2 - \text{Cl}_{P_2}(G)) \subset \text{Cl}_{Q_1}(\text{Int}_{P_1}(F^+(X_2 - \text{Cl}_{P_2}(G)))). \text{ So}$$

$$X_1 - F^-(\text{Cl}_{P_2}(G)) \subset \text{Cl}_{Q_1}(\text{Int}_{P_1}(X_1 - F^-(\text{Cl}_{P_2}(G)))) = \text{Cl}_{Q_1}(X_1 -$$

$$- \text{Cl}_{P_1}(F^-(\text{Cl}_{P_2}(G)))) = X_1 - \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^-(\text{Cl}_{P_2}(G)))) \subset$$

$$\subset X_1 - \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^-(G))).$$

Thus we have

$$\text{Int}_{Q_1}(\text{Cl}_{P_1}(F^-(G))) \subset F^-(\text{Cl}_{P_2}(G)).$$

Let $U = \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^-(G)))$. As F is $(2,1) - H$. l. a. c. and G is $Q_2 -$ open then $x \in U$. So U is $Q_1 -$ open, $x \in U$ and $F(U) \cap \text{Cl}_{P_2}(G) \neq \emptyset$. $\forall u \in U$, Thus F is $(2,1) -$ u. w. c. at x .

Theorem 2. If the multifunction $F: (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is $(1,2) -$ l. q. c. and $(2,1) - H$. u. a. c. then F is $(2,1) -$ u. w. c.

Proof. Let $x \in X_1$ and G be any $Q_2 -$ open set such that $F(x) \subset G$. Then $(X_2 - \text{Cl}_{P_2}(G))$ is $P_2 -$ open and by Lemma 2, $F^-(X_2 - \text{Cl}_{P_2}(G))$ is $(1,2) -$ semi open. By [1, Theorem 1.1] we have

$$F^-(X_2 - \text{Cl}_{P_2}(G)) \subset \text{Cl}_{Q_1}(\text{Int}_{P_1}(F^-(X_2 - \text{Cl}_{P_2}(G)))).$$

So

$$\begin{aligned} X_1 - F^+(\text{Cl}_{P_2}(G)) &\subset \text{Cl}_{Q_1}(\text{Int}_{P_1}(X_1 - F^+(\text{Cl}_{P_2}(G)))) = \\ &= \text{Cl}_{Q_1}(X_1 - \text{Cl}_{P_1}(F^+(\text{Cl}_{P_2}(G)))) = X_1 - \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^+(\text{Cl}_{P_2}(G)))) \subset \\ &\subset X_1 - \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^+(G))). \end{aligned}$$

Thus we have

$$\text{Int}_{Q_1}(\text{Cl}_{P_1}(F^+(G))) \subset F^+(\text{Cl}_{P_2}(G)).$$

Let $U = \text{Int}_{Q_1}(\text{Cl}_{P_1}(F^+(G)))$. As F is $(2, 1) - H. u. a. c.$ and G is Q_2 -open then $x \in U$. So, U is Q_1 -open, $x \in U$ and $F(U) \subset \text{Cl}_{P_2}(G)$. Thus F is $(2, 1) - u. w. c.$ at x .

Corollary 1. (Popa [15]). *If the multifunction $F : (X, P) \rightarrow (Y, Q)$ is u. q. c. and H. la. a. c., then F is l. w. c.*

Corollary 2 (Popa [15]). *If the multifunction $F : (X, P) \rightarrow (Y, Q)$ is l. q. c. and H. u. a. c., then F is u. w. c.*

Theorem 3. *Suppose (X_2, P_2, Q_2) is Q_2 regular with respect to P_2 and $F(x_0)$ strictly (Q_2, P_2) paracompact, then the multifunction $F : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is $(2, 1) u. w. c.$ at x_0 if and only if $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is u. s. c.*

Proof. Suppose F is $(2, 1) - u. w. c.$ at $x_0 \in X_1$. Let $G \subset X_2$ a Q_2 -open set containing $F(x_0)$. The space (X_2, P_2, Q_2) being Q_2 regular with respect to P_2 , for every $y_i \in F(x_0)$ by [16, Proposition 5. 2] there is a Q_2 -open set D_i such that $y_i \in D_i \subset \text{Cl}_{P_2}(D_i) \subset G$. So, we have

$$F(x_0) \subset U\{D_i : y_i \in F(x_0)\} \subset U\{\text{Cl}_{P_2}(D_i) : y_i \in F(x_0)\} \subset G.$$

$F(x_0)$ being strictly (Q_2, P_2) paracompact there is a family $\{A_j : j \in J\}$ of Q_2 -open sets such that $A_j \subset D_i$ for some j and $\{A_j : j \in J\}$ is a P_2 -locally finite covering of $F(x_0)$. If I denotes the set of all admissible values of i , we have

$$F(x_0) \subset U\{A_j : j \in J\} \subset \{\text{Cl}_{P_2}(D_i) : i \in I\} \subset G.$$

Let $A = U\{A_j : j \in J\}$, then $\text{Cl}_{P_2}(A) = U\{\text{Cl}_{P_2}(A_j) : j \in J\}$.

The multifunction F being $(2, 1) - u. w. c.$ at x_0 , A Q_2 -open and $F(x_0) \subset A$, there is a Q_1 -open set $U \subset X_1$ containing x_0 such that $F(U) \subset \text{Cl}_{P_2}(A) \subset G$. This shows that $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is u. s. c. in x .

The reciprocity is obvious.

Lemma 3. *Let (X, P, Q) be a bitopological space Q regular with respect to P and $A \subset X$, then for every Q -open set D which intersects A , there exists a Q -open set D_A so that $A \cap D_A \neq \emptyset$ and $\text{Cl}_P(D_A) \subset D$.*

Proof. Let $x \in A \cap D$, then $x \in (X - D)$ and $(X - D)$ is Q -closed. Q being regular with respect to P there exists a Q -open set U and a P -open set V disjoint from U such that $x \in U$ and $(X - D) \subset V$. From $U \cap V = \emptyset$ there follows that $U \cap \text{Cl}_Q(V) = \emptyset$ and thus $x \in \text{Cl}_Q(V)$. Let $D_A = X - \text{Cl}_Q(V)$, then D_A is Q -open and $x \in D_A$, so $A \cap D_A \neq \emptyset$. On the other hand $\text{Cl}_P(D_A) = \text{Cl}_P(X - \text{Cl}_Q(V)) \subset \text{Cl}_P(X - V) = X - V \subset D$.

Theorem 4. *If (X_2, P_2, Q_2) is Q_2 -regular with respect to P_2 , then the multifunction $F : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is $(2, 1) - l. w. c.$ at x_0 if and only if $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is l. s. c.*

Proof. Suppose F is $(2, 1) - l. w. c.$ at x_0 . Let $G \subset X_2$ be a Q_2 -open set such that $F(x_0) \cap G \neq \emptyset$. As (X_2, P_2, Q_2) is Q_2 -regular with respect to P_2 and $F(x_0) \cap G \neq \emptyset$, there exists a Q_2 -open set D such that $D \cap F(x_0) \neq \emptyset$ and $\text{Cl}_{P_2}(D) \subset G$. The multifunction F being $(2, 1) - l. w. c.$ at x_0 , D Q_2 -open and $D \cap F(x_0) \neq \emptyset$, there exists a Q_1 -open set U containing x_0 such that $F(u) \cap \text{Cl}_{P_2}(D) \neq \emptyset$ for every $u \in U$, thus $F(u) \cap G \neq \emptyset$ for every $u \in U$. This shows that $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is l. s. c. in x_0 .

The reciprocity is obvious.

Corollary 3 (Popa, [13]). *For the multifunction $F : (X, P) \rightarrow (Y, Q)$ with Y T_3 space and for which $F(x)$ is strictly paracompact set, the concept of u. w. c. multifunction in x coincides with the concept of u. s. c. multifunction.*

Corollary 4 (Popa, [12]). *For the multifunction $F : (X, P) \rightarrow (Y, Q)$, with Y T_3 space, the concept of l. w. c. multifunction in x coincides with the concept of l. s. c. multifunction in x .*

Definition 8 [8]. *Let $F : (X_1, P_1, P_2) \rightarrow (Y, Q_1, Q_2)$ be a multifunction :*

a) F is $(P_1, Q_1) - S$ upper almost continuous (S. u. a. c.) with respect to Q_2 at x_0 if for every Q_1 -open set V containing $F(x_0)$, there exists a P_1 -open set U containing x_0 such that $F(U) \subset \text{Int}_{Q_1}(\text{Cl}_{Q_2}(V))$

Similarly we define $(P_2, Q_2) - S$ upper almost continuity with respect to Q_1 at $x_0 \in X_1$.

b) F is $(P_1, Q_1) - S$ lower almost continuous (S. l. a. c.) with respect to Q_2 at x_0 if for every Q_1 -open set V with $F(x_0) \cap V \neq \emptyset$ there exists a P_1 -open set U containing x_0 such that $F(u) \cap \text{Int}_{Q_1}(\text{Cl}_{Q_2}(U)) \neq \emptyset$ for every $u \in U$.

Similarly we define $(P_2, Q_2) - S$ lower almost continuity with respect to Q_1 at $x_0 \in X_1$.

Corollary 5. *If (X_2, P_2, Q_2) is Q_2 regular with respect to P_2 , then the multifunction $F : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is $(Q_1, Q_2) - S. l. a. c.$ with respect to P_2 at x_0 if and only if $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is l. s. c. at x_0 .*

Corollary 6. (Mukherjee and Ganguly, [8]). *Suppose (X_2, P_2, Q_2) is Q_2 -regular with respect to P_2 and $F(x_0)$ strictly (Q_2, P_2) paracompact, then the multifunction $F : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is $(Q_1, Q_2) - S. u. a. c.$ with respect to P_2 at $x_0 \in X_1$ if and only if the multifunction $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is u. s. c. at x_0 .*

Definition 9 [9]. A single valued mapping $f : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is said to be pairwise continuous if the induced functions $f : (X_1, P_1) \rightarrow (X_2, P_2)$ and $f : (X_1, Q_1) \rightarrow (X_2, Q_2)$ are continuous.

Corollary 7. *Let $f : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ a single valued mapping with (X_2, P_2, Q_2) pairwise regular. Then f is weakly pairwise continuous if and only if it is pairwise continuous.*

Theorem 5. *Suppose (X_2, P_2, Q_2) is Q_2 regular with respect to P_2 and $F(x_0)$ strictly (Q_2, P_2) paracompact. If the multifunction $F : (X_1, P_1, Q_1) \rightarrow$*

$\rightarrow (X_2, P_2, Q_2)$ is $(1, 2)$ -l. q. c. and $(2, 1)$ -H. u. a. c. at x_0 , then $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is s. c. s. in x_0 .

Proof. Follows from Theorems 2 and 3.

Theorem 6. Suppose (X_2, P_2, Q_2) is Q_2 regular with respect to P_2 . If the multifunction $F : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ is $(1, 2)$ -u. q. c. and $(2, 1)$ -H. l. a. c. at x_0 , then $F : (X_1, Q_1) \rightarrow (X_2, Q_2)$ is l. s. c. at x_0 .

Proof. Follows from Theorems 1 and 4.

Corollary 8. (Ewert, [3]). Let X be a topological space and (Y, T_1, T_2) be a bitopological space in which T_2 is regula with respect to T_1 .

a) If a multifunction $F : X \rightarrow (Y, T_1, T_2)$ is T_2 -H. l. a. c. and T_1 -u. q. c. at x_0 , then F is T_2 -l. s. c. at x_0 .

b) If for each $x \in X$ the set $F(x)$ is T_2 -compact, F is T_2 -H. u. a. c. and T_1 -l. q. c. at x_0 , then F is T_2 -u. s. c. at x_0 .

Corollary 9. (Lipski, [6]). Let Y be a regular topological space. If a multifunction $F : (X, T_1, T_2) \rightarrow Y$ with $F(x)$ a compact set $\forall x \in X$, is $(1, 2)$ u. q. c. and $(2, 1)$ -H. l. a. c., then F is T_2 -l. s. c.

Corollary 10. (Lipski, [6]). Let Y be a regular topological space. If a multifunction $F : (X, T_1, T_2) \rightarrow Y$ is $(1, 2)$ -l. q. c. and $(2, 1)$ -H. u. a. c., then F is T_2 -u. s. c.

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