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QUADRATURE RULES OBTAINED BY MEANS OF INTERPOLATORY LINEAR POSITIVE OPERATORS

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1. The linear positive operators of interpolatory type enable us to generate quadrature rules, with positive coefficients, for weightedintegrals, having different degrees of exactness.

Let (L_m) be a sequence of linear positive operators, where $L_m : C[a, b] \to C[a, b]$ is defined, for any $f \in C[a, b]$, by a formula of the following form

$$(L_m f)(x) := \sum_{k=0}^{m} q_{m,k}(x) f(x_{m,k}), \tag{1}$$

where the notes $x_{m,k}$ are distinct points of the interval $[a, b]$, while $q_{m,k}$ are non-negative polynomials on $[a, b]$ for any $k = 0(1)m$ and $m \in \mathbb{N}$.

The operator L_m is called interpolatory because the values of $L_m f$ are expressed by means of the values of the function f at a finite number of points.

It is known that according to the theorem of Bohman-Korovkin, the sequence $(L_m f)$ converges uniformly to f on $[a, b]$, for any $f \in C[a, b]$, if and only if such a convergence occur for a triplet of „test functions" from $C[a, b]$ forming a Korovkin system. The three monomials e_0, e_1, e_2, where $e_j(x) = x^j$ $(j = 0, 1, 2)$, represent such a system.

2. By using the operator L_m defined at (1) we can construct a quadrature formula, for a weighted integral, of the following form

$$J(w; f) = \int_a^b w(x) f(x)\, dx = \sum_{k=0}^{m} A_{m,k} f(x_{m,k}) + R_m(w; f), \tag{2}$$

where w is a positive weight function, while the weight coefficients are given by the formula: $A_{m,k} = J(w; q_{m,k})$.

It should be noticed that, in general, the degree of exactness of formula (2) is at most one, since a linear positive operator can preserve only the constants or the linear functions. Consequently, the degree of exactness of the quadrature formula (2) is expected to be at most $N = 1$. In such a case, if we assume that $f \in C^2[a, b]$, we can apply the well-known theorem of Peano and obtain the following integral representation of the remainder

$$(3)\qquad R_m(w\,;f) = \int_a^b G_m(t)\, f''(t)\, \mathrm{d}t,$$

where G_m is the Peano kernel, defined by

$$(4)\quad G_m(t) = R_m(w\,;\,(x-t)_+)_x = \int_a^b w(x)\,(x-t)_+\,\mathrm{d}x - \sum_{k=0}^{m} A_{m,k}(x_{m,k}-t)_+ .$$

If G_m does not change sign on (a, b), then we can find at once that

$$(5)\qquad R_m(w\,;f) = \frac{1}{2}\, R_m(w\,;e_2)\, f''(\xi),\ a < \xi < b.$$

According to a result of T. Popoviciu [2], we can give a representation for this remainder in terms of divided differences of second-order, namely

$$R_m(w\,;f) = R_m(w\,;e_2)\cdot [t_1, t_2, t_3\,;f],$$

if we assume that $R_m(w\,;f) \neq 0$ whenever f is a convex function of first-order, that is having different from zero all the divided differences of second-order on any triplet of distinct points from $[a, b]$.

3. For illustration, let us consider first the case of a Bernstein-type operator considered first, in 1969, in the paper [4], $B_m^{\alpha,\beta}$ defined by

$$(6)\qquad (B_m^{\alpha,\beta} f)\,(x) := \sum_{k=0}^{m} p_{m,k}(x)\, f\left(\frac{k+\alpha}{m+\beta}\right),$$

where $0 \leqslant \alpha \leqslant \beta$ and

$$p_{m,k}(x) := \binom{m}{k} x^k (1-x)^{m-k}.$$

By using these operators, taking $[a, b] = [0, 1]$ and $w\,(x) = 1$ on $[0, 1]$, we can obtain the following quadrature formula

$$(7)\qquad \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{m+1}\sum_{k=0}^{m} f\left(\frac{k+\alpha}{m+\beta}\right) + R_m^{\alpha,\beta}(f).$$

We find that

$$R_m^{\alpha,\beta}(\mathrm{e}_0) = 0,\ R_m^{\alpha,\beta}(\mathrm{e}_1) = \frac{\beta-2\alpha}{2(m+\beta)},\ R_m^{\alpha,\beta}(\mathrm{e}_2) = -\frac{m+6\alpha(m+\alpha)-2\beta\,(2m+\beta)}{6(m+\beta)^2}.$$

One observes that we can increase the degree of exactness of formula (7) to $N = 1$; indeed, if we select $\beta = 2\alpha$, then we have

$$(8)\qquad R_m^{\alpha,2\alpha}(\mathrm{e}_2) = -\frac{m-2\alpha\,(m+\alpha)}{6(m+2\alpha)^2}.$$

Consequently, according to formulas (5), (7) and (8) we can write the following quadrature formula

$$(9)\qquad \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{m+1}\sum_{k=0}^{m} f\left(\frac{k+\alpha}{m+2\alpha}\right) - \frac{m-2\alpha\,(m+\alpha)}{12(m+2\alpha)^2}\, f''(\xi).$$

In the special case $\alpha = 0$ formula (9) reduces to the following formula

$$(10)\qquad \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{m+1}\sum_{k=0}^{m} f\left(\frac{k}{m}\right) - \frac{1}{12\,m}\, f''(\xi),$$

which corresponds to the Bernstein original operator B_m.

In a pape published in 1983 by the first author [6], there was constructed, by a probabilistic method, a Bernstein-type linear positive operator $B_{m,r}^{\alpha,\beta}$ defined by

$$(B_{m,r}^{\alpha,\beta} f)\,(x) := \sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)\, f\left(\frac{k+\alpha}{m+\beta}\right) + xf\left(\frac{k+r+\alpha}{m+\beta}\right)\right],$$

where $0 \leqslant \alpha \leqslant \beta$, r being a given non-negative integer, while m represents a natural number such that $m > 2r$. The fundamental Bernstein polynomials are given by

$$p_{m-r,k}(x) := \binom{m-r}{k} x^k (1-x)^{m-r-k}.$$

In the case $\alpha = \beta = 0$ we have the operators $B_{m,r}$ defined by

$$(B_{m,r} f)\,(x) := \sum_{k=0}^{m-r} p_{m-r,k}\,(x)\left[(1-x)\, f\left(\frac{k}{m}\right) + xf\left(\frac{k+r}{m}\right)\right]$$

and the epproximation formula

$$f(x) = (B_{m,r} f)\,(x) + (R_{m,r} f)\,(x)$$

has the degree of exactness $N = 1$.

For the remainder of this last formula the following expression has been given:

$$(R_{m,r} f)(x) = -\frac{x(1-x)}{m^2}\Big\{(m-r)\sum_{k=0}^{m-r-1} p_{m-r-1,k}(x)\Big((1-x)\cdot\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right] + x\left[x, \frac{k+r}{m}, \frac{k+r+1}{m}; f\right]\Big) +$$

$$+ r^2 \sum_{k=0}^{m-r} p_{m-r,k}(x)\left[x, \frac{k}{m}, \frac{k+r}{m}; f\right]\Big\} = -\left[1 + \frac{r(r-1)}{m}\right]\frac{x(1-x)}{2\cdot m} f''(\xi).$$

By using this operator we can construct the following quadrature formula [5]:

$$\int_0^1 f(x)\,dx = \frac{1}{(m-r+1)(m-r+2)}\left[\sum_{k=0}^{r-1}(m-r-k+1) f\left(\frac{k}{m}\right) + (m-2r+2)\sum_{k=r}^{m-r} f\left(\frac{k}{m}\right) + \sum_{k=m-r+1}^{m}(k-r+1) f\left(\frac{k}{m}\right)\right] - \frac{1}{12\,m}\left[1 + \frac{r(r-1)}{m}\right] f''(\xi),$$

which in the case $r = 1$ reduces to formula (10).

4. It is a surprising fact that by using an approximation formula by means of a special linear positive operator, which has the degree of exactness of only $N = 0$, we can construct a quadrature formula having a maximum degree of exactness: $2m + 1$.

Indeed, let us take $[a, b] = [-1, 1]$ and the weight function $w(x) := (1-x)^{-1/2}$, and consider the approximation formula

$$f(x) = (F_{2m+1} f)(x) + (R_{2m+1}(f)(x), \tag{11}$$

where F_{2m+1} is the operator of Hermite-Fejér, defined by

$$(F_{2m+1} f)(x) := \sum_{k=0}^{m}(1 - x_k x)\left(\frac{T_{m+1}(x)}{(m+1)(x-x_k)}\right)^2 f(x_k),$$

where $T_{m+1}(x) := \cos[(m+1)\arccos x]$ and the nodes are the zeros of this orthogonal polynomial of Tchebyshev of first kind. By using formula (11) we arrive at the Gauss-Mehler quadrature formula

$$\int_{-1}^{1}\frac{f(x)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{m+1}\sum_{k=0}^{m} f\left(\cos\frac{2k+1}{2m+2}\pi\right) + \rho_m(f), \tag{12}$$

where, if we assume that $f \in C^{2m+2}(-1, 1)$, we have

$$\rho_m(f) = \frac{\pi}{2^{2m+1}}\cdot\frac{f^{(2m+2)}(\xi)}{(2m+2)!}.$$

For proving that this quadrature formula can be obtained by using the operator F_{2m+1}, we can start from the Hermite interpolation formula using double nodes:

$$f(x) = \sum_{k=0}^{m} g_k(x) f(x_k) + \sum_{k=0}^{m} h_k(x) f'(x_k) + u^2(x)[x, x_0, x_0, x_1, x_1, \ldots, x_m, x_m; f],$$

where

$$u(x) = (x - x_0)(x - x_1)\ldots(x - x_m),$$

$$g_k(x) = \left[1 - (x - x_k)\frac{u''(x_k)}{u'(x_k)}\right]p_k^2(x),\quad h_k(x) = (x - x_k)\,p_k^2(x),\quad p_k^2(x) = \frac{u(x)}{(x - x_k)\,u'(x_k)}.$$

By choosing $u(x) = 2^{-m}\cdot T_{m+1}(x)$, we obtain

$$g_k(x) = \frac{1}{(m+1)^2}\cdot(1 - x_k x)\left(\frac{T_{m+1}(x)}{x - x_k}\right)^2,$$

$$h_k(x) = \frac{1 - x_k^2}{(m+1)^2}\cdot\frac{T^2_{m+1}(x)}{x - x_k}.$$

On the other hand, it is known that

$$\int_{-1}^{1}\frac{g_k(x)}{\sqrt{1-x_k^2}}\,dx = \frac{\pi}{m+1},\quad \int_{-1}^{1}\frac{h_k(x)}{\sqrt{1-x_k^2}}\,dx = 0 \text{ (ortogonality)}.$$

Concerning the remainder, it is easy to see that we can write successively

$$\rho_m(f) = \int_{-1}^{1}\frac{u^2(x)}{\sqrt{1-x^2}}[x, x_0, x_0, x_1, x_1, \ldots, x_m x_m; f]\,dx =$$

$$= \frac{1}{(2m+2)!} f^{(2m+2)}(\xi)\cdot\frac{1}{4m}\int_{-1}^{1}\frac{T^2_{m+1}(x)}{\sqrt{1-x_k^2}}\,dx = \frac{\pi}{2^{2m+1}}\cdot\frac{f^{(2m+2)}(\xi)}{(2m+2)!}.$$

5. Now let us consider the Hermite-Fejér operator $F^{(\alpha,\beta)}_{2m+1}$ using as nodes the zeros $x_k = x_{m,k}$ of the Jacobi polynomials

$$J^{(\alpha,\beta)}_{m+1}(x) = \frac{(-1)^{m+1}(1-x)^{-\alpha}(1+x)^{-\beta}}{2^{m+1}\cdot(m+1)!}\left[(1-x)^{m+1+\alpha}(1+x)^{m+1+\beta}\right]^{(m+1)}.$$

We have

$$(F_{2m+1}^{(\alpha,\beta)} f)(x) := \sum_{k=0}^{m} g_k(x)\, p_k^2(x)\, f(x_{m,k}),$$

where

$$g_k(x) = g_{m,k}^{(\alpha,\beta)}(x) = 1 - \frac{\alpha - \beta + (\alpha + \beta + 2)x_k}{1 - x_k^2}(x - x_k)$$

and

$$p_k(x) = p_{m,k}^{(\alpha,\beta)}(x) = \frac{J_{m+1}^{(\alpha,\beta)}(x)}{(x - x_{m,k})\, J_{m+1}^{\prime(\alpha,\beta)}(x_{m,k})}$$

If the Hermite-Fejér operator $F_{2m+1}^{(\alpha,\beta)}$ is applied to the function $\varphi_x(t) = (x-t)^2$, it follows that for any α and β greater than -1 we have [1]:

$$(F_{2m+1}^{(\alpha,\beta)} \varphi_x)(x) = \sigma_{2m+1}^{(\alpha,\beta)} [J_{m+1}^{(\alpha,\beta)}(x)]^2,$$

where

$$\sigma_{2m+1}^{(\alpha,\beta)} := 8\frac{\Gamma(\alpha + 2)\,\Gamma(\beta + 2)\cdot(m + \alpha)!\,\Gamma(m + \alpha + \beta + 2)}{\Gamma(\alpha + \beta + 4)\cdot\Gamma(m + \alpha + 2)\,\Gamma(m + \beta + 2)} = 0(1)\ (m \to \infty).$$

According to a well-known result of Szegö [8] the Hermite-Fejér operators are positive for all $m \in \mathbb{N}$ iff $\max(\alpha, \beta) \leqslant 0$. Therefore the Bohman-Korovkin theorem may be applied and we find that

$$\lim_{m \to \infty} F_{2m+1} f = f,$$

uniformly on $[-1, 1]$.

In the book [7] there are mentioned sufficient conditions for the uniform convergence of $(F_{2m+1}^{(\alpha,\beta)} f)$ to $f \in C[-1, 1]$ in the case when $\max(\alpha, \beta) \geqslant 0$. At the same time one discussed the possibility of reducing this problem to the behavior of it at the end points of the interval $[-1,1]$.

By using the operator $F_{2m+1}^{(\alpha,\beta)}$, we can construct a quadrature formula, using the Jacobi nodes, of the following form

(13) $$\int_{-1}^{1} (1-x)^{\alpha} (1 + x)^{\beta} f(x)\, dx = \sum_{k=0}^{m} A_{m,k}^{(\alpha,\beta)} \cdot f(x_{m,k}^{(\alpha,\beta)}) + R_m^{(\alpha,\beta)}(f),$$

which has the maximum degree of exactness $N = 2m + 1$.

The coefficients can be expressed by the formula

(14) $$A_{m,k}^{(\alpha,\beta)} = \frac{2^{\alpha+\beta}(2m + \alpha + \beta + 2)\,\Gamma(m + \alpha + 1)\,\Gamma(m + \beta + 1)}{(m + 1)!\,\Gamma(m + \alpha + \beta + 2)\, J_m^{(\alpha,\beta)}(x_k)\, J_{m+1}^{\prime(\alpha,\beta)}(x_k)},$$

the nodes being the zeros of $J_{m+1}^{(\alpha,\beta)}(x)$.

The remainder is given by the formula

(15) $$\mathbb{R}_m^{(\alpha,\beta)}(f) = 2^{2m+\alpha+\beta+3} \frac{\Gamma(m+2)\,\Gamma(m+\alpha+2)\,\Gamma(m+\beta+2)\,\Gamma(m+\alpha+\beta+2)}{\Gamma(2m+3)\,\Gamma(2m+\alpha+\beta+3)\,\Gamma(2m+\alpha+\beta+4)} f^{(2m+2)}(\xi).$$

In the special case $\alpha = \beta = -1/2$ the quadrature formula given at (13), (14), (15) reduces to the Gauss–Mehler formula (12).

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