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INTEGRAL AND DISCRETE INEQUALITIES

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1. Introduction. In [8] we have proved some integral inequalities showing that the inequalities are valid for a sequence of integral sums with norm tending to zero. In this paper, starting from some integral inequalities, we prove discrete versions.

To avoid complications related to the integrability, we suppose all the functions which appear in what follows to be continuous. The following results were considered in [8]:

Theorem A. *If the function $f : [a, b] \to \mathbb{R}$ is Jensen convex, $h : [a, b] \to \mathbb{R}$ is positive and symmetric with respect to $(a + b)/2$, then :*

$$f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)\, h(x)\, dx \Big/ \int_a^b h(x)\, dx \leqslant \frac{f(a) + f(b)}{2}.$$

Theorem B. *If the function $f : [a, b] \to [c, d]$ is increasing and Jensen convex and $g, h : [c, d] \to [0, \infty)$ are such that g/h is increasing, then :*

$$\int_a^b g(f(x))\, dx \Big/ \int_a^b h(f(x))\, dx \leqslant \int_{f(a)}^{f(b)} g(x)\, dx \Big/ \int_{f(a)}^{f(b)} h(x)\, dx. \tag{1}$$

Remark 1. The first theorem was proved by L. Fejér in [2] and for $h(x) = 1$ it gives the inequality of Hermite – Hadamard. The inequality (1) was proposed as a problem by A. Lupaş in [5] for $g(x)=x^2$ and $h(x)=x$. It was proved by L. Daia in [1] for $g(x) = x^r$ and $h(x) = x^s$ with $r > s$. In the form (1) it was given by I. Gavrea in [3] but under the assumption of differentiability of f. We have shown in [8] that the inequality is valid without this last condition.

The following result was given by J. Kolumban and C. Mocanu in [4].

Theorem C. *If the functions $f, g, h : [a, b] \to \mathbb{R}$ are positive, g is increasing and differentiable, $g(a) > 0$ and :*

$$\int_a^x f^q(t)\, h(t)\, dt \leqslant \int_a^x g^q(t)\, h(t)\, dt,\ \forall x \in [a, b]$$

then for $0 < p < q$:

$$\int_a^x f^p(t)\, h(t)\, dt \leqslant \int_a^x g^p(t)\, h(t)\, dt, \quad \forall x \in [a, b].$$

2. Finite differences. For a sequence $(x_k)_{k=1}^n$, we consider the finite differences of order one:

$$\Delta_p^1 x_k = x_{k+p} - x_k, \quad 1 \leqslant k < k + p \leqslant n$$

and of order two:

$$\Delta_{pq}^2 x_k = x_{k+p+q} - (1 + q/p)\, x_{k+p} + (q/p)\, x_k, \; 1 \leqslant k < k+p < k + p + q \leqslant n$$

We denote simply $\Delta_1^1 = \Delta^1$ and $\Delta_{11}^2 = \Delta^2$.

A sequence $(x_k)_{k=1}^n$ is increasing if $\Delta^1 x_k \geqslant 0$ for $1 \leqslant k \leqslant n - 1$, but this is equivalent with the condition:

$$\Delta_p^1 x_k \geqslant 0, \text{ for } 1 \leqslant k < k + p \leqslant n$$

as:

$$\Delta_p^1 x_k = \sum_{i=1}^{p} \Delta^1 x_{k+i-1}. \tag{2}$$

Analogously, the sequence $(x_k)_{k=1}^n$ is said to be convex if $\Delta^2 x_k \geqslant 0$ for $1 \leqslant k \leqslant n - 2$ and this is equivalent with:

$$\Delta_{pq}^2 x_k \geqslant 0, \text{ for } 1 \leqslant k < k + p < k + p + q \leqslant n.$$

because we have:

Lemma 1. *For every k, p and q:*

$$\Delta_{pq}^2 x_k = \sum_{i=1}^{q} i \cdot \Delta^2 x_{k+p+q-i-1} + (q/p) \sum_{j=1}^{p-1} j\, \Delta^2 x_{k+j-1} \tag{3}$$

hold.

Proof. We have:

$$\Delta_{pq}^2 x_k = \Delta_q^1 x_{k+p} - (q/p)\, \Delta_p^1 x_k$$

and using (2):

$$\Delta_{pq}^2 x_k = \sum_{i=1}^{q} \Delta^1 x_{k+p+i-1} - (q/p) \sum_{j=1}^{p} \Delta^1 x_{k+j-1}. \tag{4}$$

Applying Abel's identities:

$$\sum_{i=1}^{q} y_i = \sum_{i=1}^{q-1} i \cdot \Delta^1 y_{q-i} + q \cdot y_1$$

respectively

$$\sum_{j=1}^{p} z_j = p \cdot z_p - \sum_{j=1}^{p-1} j\, \Delta^1 z_j$$

for the two sums of (8), we get:

$$\Delta_{pq}^2 x_k = \sum_{i=1}^{q-1} i \cdot \Delta^2 x_{k+p+q-i-1} + q \cdot \Delta^1 x_{k+p} -$$

$$- (q/p) \left(p\, \Delta^1 x_{k+p-1} - \sum_{j=1}^{p-1} j \cdot \Delta^2 x_{k+j-1} \right)$$

thus (3).

Remark 2. Relation (3) is similar with that given by T. Popoviciu in [7] for divided differences.

3. Discrete inequalities. We begin with a discrete version of Fejér's inequality. We say that the sequence $(p_i)_{i=1}^n$ is symmetric if:

$$p_i = p_{n-i+1} \text{ for } 1 \leqslant i \leqslant n.$$

Theorem 1. *If the sequence $(x_i)_{i=1}^n$ is convex and $(p_i)_{i=1}^n$ is symmetric and positive, then:*

$$(x_{[(n+1)/2]} + x_{[(n+2)/2]})/2 \leqslant \sum_{i=1}^{n} x_i\, p_i \Big/ \sum_{i=1}^{n} p_i \leqslant (x_1 + x_n)/2 \tag{5}$$

where $[a]$ denotes the integer part of a.

Proof. As $\Delta_{i-1,n-i}^2 x_1 \geqslant 0$, we have:

$$(n - 1)\, x_i \leqslant (i - 1)\, x_n + (n - i)\, x_1.$$

Putting $n - i + 1$ instead i we get:

$$(n - 1)\, x_{n-i+1} \leqslant (n - i)\, x_n + (i - 1)\, x_1$$

and by addition:

$$x_i + x_{n-i+1} \leqslant x_1 + x_n.$$

Multiplying by $p_i = p_{n-i+1}$ and adding for $i = 1, \ldots, n$ we get the second part of (5). For the first part we consider separately the case of n odd or n even. So, if $n = 2m + 1$, as $\Delta_{m-i+1,\, m-i+1}^2 x_i \geqslant 0$, we have:

$$2 \cdot x_{m+1} \leqslant x_i + x_{2m+2-i}.$$

Multiplying by $p_i = p_{2m+2-i}$ and adding for $i = 1, \ldots, 2m+1$ we get:

$$\sum_{i=1}^{2m+1} x_i p_i / \sum_{i=1}^{2m+1} p_i \geq x_{m+1} = (x_{[(2m+2)/2]} + x_{[(2m+3)/2]})/2.$$

For $n = 2m$, we have:

$$(m-i)\,\Delta^2_{m-i,\,m-i+1}\, x_i + (m-i+1)\,\Delta^2_{m-i+1,\,m-i}\, x_i \geq 0$$

hence:

$$x_m + x_{m+1} \leq x_i + x_{2m-i+1}.$$

Multiplying by $p_i = p_{2m-i+1}$ and adding for $i = 1, \ldots, 2m$, we obtain:

$$\sum_{i=1}^{2m} x_i p_i / \sum_{i=1}^{2m} p_i \geq (x_m + x_{m+1})/2 = (x_{[(2m+1)/2]} + x_{[(2m+2)/2]})/2$$

Remark 3. For $p_i = 1$ $(i = 1, \ldots, n)$ we get a discrete variant of Hermite-Hadamard inequality. On the other hand, inequality (5) can be used for the proof of Fejér's integral inequality.

Passing to theorem B, we can see that inequality (1) holds if and only if for every natural n, denoting:

$$x_i = a + (i-1)(b-a)/n, i = 1, \ldots, n+1$$

we have the inequality:

$$(6) \qquad \sum_{i=1}^{n} g(f(x_i)) / \sum_{i=1}^{n} h(f(x_i)) \leq \sum_{i=1}^{n} g(f(x_i))\,\Delta^1 f(x_i) / \sum_{i=1}^{n} h(f(x_i))\,\Delta^1 f(x_i)$$

But we can prove a much stronger result which generalizes also Cauchy's inequality and Čebyšev's inequality (see [6]). We say that the sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ are synchrone if:

$$(a_i - a_j)(b_i - b_j) \geq 0,\ 1 \leq i, j \leq n.$$

Theorem 2. *If the sequences $(y_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ are strictly positive and $(x_i/y_i)_{i=1}^n$ and $(p_i/q_i)_{i=1}^n$ are synchrone, then:*

$$(7) \qquad \sum_{i=1}^{n} x_i p_i \sum_{i=1}^{n} y_i q_i \geq \sum_{i=1}^{n} x_i q_i \sum_{i=1}^{n} y_i p_i.$$

Proof. As:

$$(x_i/y_i - x_j/y_j)(p_i/q_i - p_j/q_j) \geq 0$$

we have:

$$x_i p_i y_j q_j - x_j q_j y_i p_i - x_i q_i y_j p_j + x_j p_j y_i q_i \geq 0$$

and adding consecutively for $i = 1, \ldots, n$ and then for $j = 1, \ldots, n$ we get (7).

Remark 4. This is a discrete variant of an integral inequality of M. Fujiwara (see [6]). For $p_i = x_i$ and $q_i = y_i$, $i = 1, \ldots, n$, we have Cauchy's inequality and for $y_i = q_i = 1$, $i = 1, \ldots, n$, we have Čebyšev's inequality.

If the sequence $(p_i)_{i=1}^{n+1}$ is convex, then the sequence $(\Delta^1 p_i)_{i=1}^n$ is increasing and taking $q_i = 1$ for $i = 1, \ldots, n$, we have the following result which also implies (6):

Consequence. If the sequence $(y_i)_{i=1}^n$ is strictly positive, $(x_i/y_i)_{i=1}^n$ is increasing and $(p_i)_{i=1}^{n+1}$ convex, then:

$$\sum_{i=1}^{n} x_i \Delta^1 p_i \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} y_i \Delta^1 p_i \sum_{i=1}^{n} x_i.$$

To prove a discrete version of theorem C we need the following:

Lemma 2. *If the sequence $(b_i)_{i=1}^n$ is positive and decreasing, then:*

$$\sum_{i=1}^{k} a_i \geq m,\ \forall k \leq n$$

implies:

$$\sum_{i=1}^{n} a_i b_i \geq m b_1.$$

Proof. Using Abel's identity, we have:

$$\sum_{i=1}^{n} a_i b_i = \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} a_i \right)(b_k - b_{k+1}) + \sum_{i=1}^{n} a_i b_n \geq$$

$$\geq m \left(\sum_{k=1}^{n-1} (b_k - b_{k+1}) + b_n \right) = m \cdot b_1.$$

Theorem 3. *If the sequences $(x_i)_{i=1}^n$ and $(z_i)_{i=1}^n$ are positive and $(y_i)_{i=1}^n$ is strictly positive and increasing, then:*

$$\sum_{i=1}^{k} x_i^q z_i \leq \sum_{i=1}^{k} y_i^q z_i,\ \forall k = 1, \ldots, n$$

implies:

$$\sum_{i=1}^{k} x_i^p z_i \leq \sum_{i=1}^{k} y_i^p z_i,\ \forall k = 1, \ldots, n$$

for $0 < p < q$.

Proof. We use Hölder's inequality :

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^r\right)^{1/r} \left(\sum_{k=1}^{n} b_k^s\right)^{1/s}, \; r > 1, \; 1/r + 1/s = 1$$

for $r = q/p$ and $s = q/(q-p)$. So :

$$\left(\sum_{i=1}^{k} x_i^p z_i\right)^q = \left(\sum_{i=1}^{k} (x_i^p z_i^{1/r}/y_i^{p/s})\, y_i^{p/s} z_i^{1/s}\right)^q \leqslant$$

$$\leqslant \left(\sum_{i=1}^{k} x_i^{rp} z_i/y_i^{pr/s}\right)^{q/r} \left(\sum_{i=1}^{k} y_i^p z_i\right)^{q/s} = \left(\sum_{i=1}^{k} x_i^q z_i/y_i^{q-p}\right)^p \cdot$$

$$\cdot \left(\sum_{i=1}^{k} y_i^p \cdot z_i\right)^{q-p} = \left(\sum_{i=1}^{k} y_i^p z_i - \sum_{i=1}^{k} z_i(y_i^q - x_i^q)/y_i^{q-p}\right)^p \cdot$$

$$\cdot \left(\sum_{i=1}^{k} y_i^p z_i\right)^{q-p} \leqslant \left(\sum_{i=1}^{k} y_i^p z_i\right)^q$$

because, by hypothesis

$$\sum_{i=1}^{k} (y_i^q - x_i^q) \cdot z_i \geqslant 0, \; \forall k$$

and $(1/y_i^{q-p})_{i=1}^n$ is decreasing, hence, by Lemma 2 :

$$\sum_{i=1}^{k} (y_i^q - x_i^q) \cdot z_i/y_i^{q-p} \geqslant 0.$$

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