REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 21 (1992) no. 1, pp. 89–100
ictp.acad.ro/jnaat

WAVELET TRANSFORM, TOEPLITZ TYPE OPERATORS AND DECOMPOSITION OF FUNCTIONS ON THE UPPER HALF-PLANE*

DING-XUAN ZHOU
(Hangzhou)

Abstract. In this paper we consider the decomposition of functions on the upper half-plane into orthogonal subspaces which are isometric to $L^2(\mathbb{R})$ by continuous wavelet transforms. A necessary and sufficient condition for such a decomposition is given. From such a decomposition by general Laguerre polynomials, we define a series of Toeplitz type operators and study the Schatten-Von Neumann classes of these operators.

1. INTRODUCTION

Let G be the affine group $\{(x,y) : y > 0,\ x \in \mathbb{R}\}$ with the group law $(x', y')(x, y) = (y'\ x + x', yy')$. It is a locally compact nonunimodular group with right Haar measure $d\mu_R(x,y) = dxdy/y$ and left Haar measure $d\mu_L(x,y) = dxdy/y^2$. It can be identified as the quotient group SL$(2,\mathbb{R})$ by SO$(2,\mathbb{R})$ [8].

We consider the representation U of G on $L^2(\mathbb{R})$ defined by

$$U_g f(x') = y^{\frac{1}{2}} f\left(\tfrac{x'-x}{y}\right). \tag{1.1}$$

By choosing a suitable function $\psi \in L^2(\mathbb{R})$, we can define an operator T^ψ from $L^2(\mathbb{R})$ to $L^{2,-2}(U)$ as

$$\left(T^\psi f\right)(g) = C_\psi^{-\frac{1}{2}} (f, U_g\psi), \tag{1.2}$$

where C_ψ is a constant depending only on ψ,

$$U = \{(x,y) : y > 0,\ x \in \mathbb{R}\},$$

$$L^{2,-2}(U) = \left\{ f(x,y) : \|f\|_{L^{2,-2}(U)} = \left(\int_U \tfrac{|f(x,y)|^2}{y^2} dxdy \right)^{\frac{1}{2}} < \infty \right\}. \tag{1.3}$$

Such an operator is called a "continuous wavelet transform" [1], [2]. It has arisen independently in mathematical analysis and in the study of signals.

If T^ψ is an isometry, we can define a subspace $A^\psi = T^\psi L^2(\mathbb{R})$ of $L^{2,-2}(U)$, which is isometric to $L^2(\mathbb{R})$. Recently, Jiang and Peng [5] have decom-

*Supported by National Science Foundation and Zhejiang Provincial Science Foundation of China.

posed $L^{2,-2}(U)$ to be the orthogonal sum $\bigoplus_{k=0}^{\infty}\left(A_k \oplus \bar{A}_k\right)$, where $A_k = A\psi^k$, $\bar{A}_k = A\bar{\psi}^k$, and $\{\psi^k, \psi^k\}_{k\in\mathbb{Z}_+}$ is a class of functions called "admissible wavelets" in $L^2(\mathbb{R})$. Then they defined the Toeplitz type operators $T_b^{k,1} = P_k M_b P_1$ with anti-analytic symbol $b(z)$ on U. The membership in the Schatten-Von Neumann class of these operators was also studied.

In this paper we give a necessary and sufficient condition for the decomposition

$$L^{2,-2}(U) = \bigoplus_{\lambda\in\Lambda} T\psi^\lambda\left(L^2(\mathbb{R})\right),$$

where $\{\psi^\lambda\}_{\lambda\in\Lambda}$ is an arbitrary class of functions in $L^2(\mathbb{R})$. For some classes of $\{\psi^\lambda\}_{\lambda\in\Lambda}$ which include the class of Jiang and Peng [5], we define the corresponding Toeplitz type operators. We also give their Schatten-Von Neumann classes S_p for $1 < p \leq \infty$. The cases $0 < p \leq 1$ will be discussed elsewhere.

2. DECOMPOSITION OF $L^{2,-2}(U)$

For $\psi \in L^2(\mathbb{R})$, the continuous wavelet transform T^ψ is defined by (1.2), where C_ψ is a constant depending only on ψ. First let us give a necessary and sufficient condition for which T^ψ is an isometry.

THEOREM 2.1. *For $\psi \in L^2(\mathbb{R})$, T^ψ is defined by (1.2), then T^ψ is an isometry from $L^2(\mathbb{R})$ onto a subspace of $L^{2,-2}(U)$, if and only if*

$$\int_0^\infty |\hat{\psi}(\omega)|^2 d\omega/\omega = \int_{-\infty}^0 |\hat{\psi}(\omega)|^2 d\omega/|\omega| < \infty.$$

Proof. If we define

$$\tilde{f}(x) = \overline{f(-x)}, \tag{2.1}$$

and

$$f_y(x) = y^{-\frac{1}{2}} f\left(\tfrac{x}{y}\right). \tag{2.2}$$

we have

$$\left(T^\psi f\right)(x,y) = C_\psi^{-\frac{1}{2}} \tilde{\psi}_y * f(x).$$

Therefore we have by taking Fourier transform for the first variable

$$\begin{aligned}
&\|T^\psi f\|_{L^{2,-2}(U)}^2 = \\
&= \int_0^\infty \tfrac{dy}{y^2}\Big(\int_{\mathbb{R}} \left|T^\psi f(x,y)\right|^2 dx\Big) \\
&= \int_0^\infty y^{-2} dy \Big(\int_{\mathbb{R}} \left|(T^\psi f)\hat{}\,(\zeta,y)\right|^2 d\zeta\Big) \\
&= \int_0^\infty y^{-2} dy C_\psi^{-1} \int_{\mathbb{R}} |\hat{\tilde{\psi}}(\zeta)|^2 |\hat{f}(\zeta)|^2 d\zeta \\
&= C_\psi^{-1}\Big(\int_0^\infty |\hat{f}(\zeta)|^2 d\zeta \int_0^\infty |\hat{\psi}(y)|^2 dy/y + \int_{-\infty}^0 |\hat{f}(\zeta)|^2 \int_{-\infty}^0 |\hat{\psi}(y)|^2 dy/|y|\Big),
\end{aligned}$$

thus we have

$$\|T^{\psi}f\|_{L^{2,-2}(U)} = \|f\|_{L^2(\mathbb{R})},$$

for any $f \in L^2(\mathbb{R})$, if and only if

$$\int_0^{\infty} |\hat{\psi}(\omega)|^2 d\omega/\omega = \int_{-\infty}^{0} |\hat{\psi}(\omega)|^2 d\omega/|\omega| = C_{\psi} < \infty.$$

The proof is complete. □

Therefore it is natural for us to define

$$AAW = \left\{ f \in L^2(\mathbb{R}) : \int_0^{\infty} |f(\omega)|^2 d\omega/\omega = \int_{-\infty}^{0} |f(\omega)|^2 d\omega/|\omega| = 1 \right\}.$$

In the case $\psi \in AAW$, we have

$$\left(T^{\psi}f\right)(x,y) = \tilde{f}_y * f(x). \tag{2.3}$$

Now we shall give a left inverse operator for T^{ψ}.

THEOREM 2.2. *For $\psi \in AAW$, let τ^{ψ} be the operator from $L^{2,-2}(U)$ to $L^2(\mathbb{R})$ defined as*

$$\left(\tau^{\psi}F\right)(x) = \int_0^{\infty} \left(\psi_y * F(\cdot, y)\right)(x)\, y^{-2} dy, \tag{2.4}$$

then τ^{ψ} is bounded, and $\tau^{\psi}T^{\psi}$ is the identity on $L^2(\mathbb{R})$.

Proof. For $F \in L^{2,-2}(U)$, we have for $\zeta \in \mathbb{R}$

$$\begin{aligned} \left|(\tau^{\psi}F)\hat{}\,(\zeta)\right|^2 &= \left| \int_0^{\infty} \hat{\psi}_y(\zeta)\left(F(\cdot,y)\right)\hat{}\,(\zeta)\, y^{-2} dy \right|^2 \\ &\leq \int_0^{\infty} |\hat{\psi}(y\zeta)|^2 y^{-1} dy \int_0^{\infty} \left|\left(F(\cdot,y)\right)\hat{}\,(\zeta)\right|^2 y^{-2} dy, \end{aligned}$$

which implies

$$\begin{aligned} \|\tau^{\psi}F\|_{L^2(\mathbb{R})} = \|(\tau^{\psi}F)\hat{}\,\|_{L^2(\mathbb{R})} &\leq \left(\int_{\mathbb{R}} \int_0^{\infty} \left|\left(F(\cdot,y)\right)\hat{}\,(\zeta)\right|^2 y^{-2} dy \right)^{\frac{1}{2}} \\ &= \left(\int_0^{\infty} y^{-2} dy \int_{\mathbb{R}} |F(x,y)|^2 dx \right)^{\frac{1}{2}} = \|F\|_{L^{2,-2}(U)}. \end{aligned}$$

Therefore we have

$$\|\tau^{\psi}\| \leq 1.$$

Now for $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\left(\tau^{\psi}T^{\psi}f\right)\hat{}\,(\zeta) &= \int_0^\infty y^{\frac{1}{2}}\hat{\psi}(y\zeta)\left(T^{\psi}f\right)\hat{}\,(\zeta)\,y^{-2}dy \\ &= \int_0^\infty y^{\frac{1}{2}}\hat{\psi}\,(y\zeta)\,y^{\frac{1}{2}}\overline{\hat{\psi}\,(y\zeta)}\hat{f}\,(\zeta)\,y^{-2}dy \\ &= \hat{f}\,(\zeta)\int_0^\infty |\hat{\psi}\,(y\zeta)\,|^2 y^{-1}dy = \hat{f}\,(\zeta)\,,\end{aligned}$$

thus $\tau^{\psi}T^{\psi}$ is the identity on $L^2(\mathbb{R})$. □

For $\psi \in AAW$, let $A^{\psi} = T^{\psi}L^2(\mathbb{R})$ be the subspace $L_{2,-2}(U)$ isometric to $L^2(\mathbb{R})$, we can define the orthogonal projection P^{ψ} from $L^{2,-2}(U)$ onto this space, then we have the following explicit formula.

THEOREM 2.3. *For ψ, A^{ψ} and P^{ψ} defined as above, we have for $F \in L^{2,-2}(U)$*

$$P^{\psi}F\,(x,y) = \int_0^\infty \tilde{\psi}_y * \psi_v * F\,(\cdot,v)\,(x)\,v^{-2}dv. \tag{2.5}$$

Proof. We can prove (2.5) from the kernel given in [3]. Here we give a new proof which is useful in the proof of our main result.

Denote

$$QF\,(x,y) = \int_0^\infty \tilde{\psi}_y * \psi_v * F\,(\cdot,v)\,(\bar{x})\,v^{-2}dv,$$

then for $F = T^{\psi}f \in A^{\psi}$, where $f \in L^2(\mathbb{R})$, we have by taking Fourier transform for the first variable

$$\begin{aligned}(QF)\,\hat{}\,(\zeta,y) &= \int_0^\infty y^{\frac{1}{2}}\overline{\hat{\psi}\,(y\zeta)}v^{\frac{1}{2}}\hat{\psi}\,(v\zeta)\,\hat{F}\,(\zeta,v)\,v^{-2}dv \\ &= \int_0^\infty y^{\frac{1}{2}}\overline{\hat{\psi}\,(y\zeta)}v^{\frac{1}{2}}\hat{\psi}\,(v\zeta)\,v^{\frac{1}{2}}\overline{\hat{\psi}\,(v\zeta)}\hat{f}\,(\zeta)\,v^{-2}dv \\ &= y^{\frac{1}{2}}\overline{\hat{\psi}\,(y\zeta)}\hat{f}\,(\zeta) \\ &= \left(\hat{\psi} * f\,(\cdot)\,\right)\hat{}\,(\zeta) = (F\,(\cdot,y))\,\hat{}\,(\zeta)\,.\end{aligned}$$

Thus we have for $F \in A\psi$

$$QF = F = P^{\psi}F.$$

On the other hand, for $F \perp A^{\psi}$, and any $g \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
&\left(QF, T^{\psi} g\right) = \\
&= \int_0^{\infty} y^{-2} dy \int_{\mathbb{R}} d\zeta \left\{ \int_0^{\infty} y^{\frac{1}{2}} \overline{\hat{\psi}(y\zeta)} v^{\frac{1}{2}} \hat{\psi}(v\zeta) F(\zeta, v) v^{-2} dv \cdot y^{\frac{1}{2}} \overline{\overline{\hat{\psi}(y\zeta)} \hat{g}(\zeta)} \right\} \\
&= \int_0^{\infty} v^{-2} dv \int_{\mathbb{R}} v^{\frac{1}{2}} \hat{\psi}(v\zeta) \overline{\hat{g}(\zeta)} \hat{F}(\zeta, v) d\zeta \\
&= \int_0^{\infty} v^{-2} dv \int_{\mathbb{R}} \overline{(T^{\psi} g)\hat{\ }(\zeta, v)} \hat{F}(\zeta, v) d\zeta \\
&= \int_0^{\infty} v^{-2} dv \int_{\mathbb{R}} \overline{(T^{\psi} g)(x, v)} F(x, v) dx \\
&= \left(T^{\psi} g, F\right)_{L^{2,-2}(U)} = 0,
\end{aligned}$$

which implies that

$$QF \perp A^{\psi}.$$

However note that $QF(x, y) = \tilde{\psi} * h(x)$, where

$$h(x) = \int_0^{\infty} \psi_v * F(\cdot, v)(x) v^{-2} dv = \tau^{\psi} F(x),$$

we know that $QF \in A^{\psi}$, therefore we must have

$$QF = 0 = P^{\psi} F.$$

Thus we must have $Q = P^{\psi}$, and we have proved (2.5). □

Now we can give the decomposition on $L^{2,-2}(U)$ by the above method. First we shall give the orthogonal condition.

Theorem 2.4. *For $\varphi, \psi \in AAW$, $A^{\psi} = T^{\psi} L^2(\mathbb{R})$ defined as above, A^{φ} is orthogonal to A^{φ}, if and only if*

$$\int_0^{\infty} \overline{\tilde{\varphi}(\omega)} \hat{\psi}(\omega) \omega^{-1} d\omega = \int_{-\infty}^{0} \overline{\hat{\psi}(\omega)} \hat{\psi}(\omega) |\omega|^{-1} d\omega = 0. \tag{2.6}$$

Proof. For $f, g \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\left(T^{\varphi} f, T^{\psi} g\right) &= \left(\hat{\varphi}_y * f, \hat{\psi}_y * g\right) \\
&= \int_0^{\infty} y^{-2} dy \int_{\mathbb{R}} y^{\frac{1}{2}} \hat{\varphi}(y\omega) \hat{f}(\omega) y^{\frac{1}{2}} \overline{\overline{\hat{\psi}(y\omega)} \hat{g}(\omega)} d\omega \\
&= \int_{\mathbb{R}} \hat{f}(\omega) \overline{\hat{g}(\omega)} \int_0^{\infty} \overline{\hat{\varphi}(y\omega)} \hat{\psi}(y\omega) y^{-1} dy \, d\omega.
\end{aligned}$$

Thus we have $\left(T^{\varphi}f, T^{\psi}g\right) = 0$ for any $f, g \in L^2(\mathbb{R})$, if and only if (2.6) holds. □

Theorem 2.5. *Let $\{\psi^\lambda : \lambda \in \Lambda\}$ be a subset of AAW, $A^{\psi^\lambda} = T^{\psi^\lambda} L^2(\mathbb{R})$ defined as above, then we have the orthogonal decomposition of $L^{2,-2}(U)$ as*

$$L^{2,-2}(U) = \bigoplus_{\lambda \in \Lambda} A^{\psi^\lambda}, \tag{2.7}$$

if and only if

$$\left\{\omega^{-\frac{1}{2}} \hat{\psi}^\lambda(\omega)\,|_{\omega \geq 0}\right\}_{\lambda \in \Lambda} \quad \text{and} \quad \left\{\,|\omega|^{-\frac{1}{2}}\, \hat{\psi}^\lambda(\omega)\,|_{\omega \leq 0}\right\}_{\lambda \in \Lambda}$$

are orthogonal bases of $L^2[0,\infty)$ and $L^2(-\infty,0]$ respectively.

Proof. As in the proof of Theorem 2.3, by taking Fourier transform about the first variable we have for $F \in L^{2,-2}(U)$

$$\begin{aligned}
\|P^{\psi^\lambda} F\|^2_{L^{2,-2}(U)} &= \int_0^\infty y^{-2} dy \int_{\mathbb{R}} \left|\left(P^{\psi^\lambda} F\right)^\wedge(\omega, y)\,\right|^2 d\omega \\
&= \int_0^\infty y^{-2} dy \int_{\mathbb{R}} \left| \int_0^\infty y^{\frac{1}{2}} \overline{\hat{\psi}(y\omega)} v^{\frac{1}{2}} \hat{\psi}^\lambda(v\omega)\, \hat{F}(\omega, v)\, v^{-2} dv \right|^2 d\omega \\
&= \int_{\mathbb{R}} \left| \int_0^\infty v^{-\frac{1}{2}} \hat{\psi}^\lambda(v\omega)\, \hat{F}(\omega, v)\, v^{-1} dv \right|^2 d\omega \\
&= \int_0^\infty \left(\int_0^\infty v^{\frac{1}{2}} \hat{\psi}^\lambda(v)\, \omega^{\frac{1}{2}} \hat{F}\big(\omega, v/\omega\big) v^{-1} dv \right)^2 d\omega \\
&\quad + \int_{-\infty}^0 \left(\int_0^\infty |v|^{-\frac{1}{2}} \hat{\psi}^\lambda(-v)\, |\omega|^{\frac{1}{2}}\, \hat{F}(\omega, -v/\omega)\, v^{-1} dv \right)^2 d\omega.
\end{aligned}$$

Sufficiency: Suppose that $\left\{\omega^{-\frac{1}{2}} \hat{\psi}^\lambda(\omega)|_{\varepsilon \geq 0}\right\}_{\lambda \in \Lambda}$ and $\left\{|\omega|^{-\frac{1}{2}}\, \psi^\lambda(\omega)|_{\omega \leq 0}\right\}_{\lambda \in \Lambda}$ are orthonormal bases of $L^2[0,\infty)$ and $L^2(-\infty,0]$ respectively, we know from Theorem 2.4 that $\{A^\psi\}_{\lambda \in \Lambda}$ is orthogonal. We also have

$$\begin{aligned}
&\sum_{\lambda \in \Lambda} \left\|P^{\psi^\lambda} F\right\|^2_{L^{2,-2}(U)} = \\
&= \int_0^\infty \sum_{\lambda \in \Lambda} \left(\int_0^\infty v^{-\frac{1}{2}} \hat{\psi}(v)\, \omega^{\frac{1}{2}} \hat{F}(\omega, v/\omega)\, v^{-1} dv \right)^2 d\omega + \\
&\quad + \int_0^{-\infty} \sum_{\lambda \in \Lambda} \left(\int_0^{-\infty} |v|^{-\frac{1}{2}}\, \hat{\psi}^\lambda(v)\, |\omega|^{\frac{1}{2}}\, \hat{F}\big(\omega, v/\omega\big)\, |v|^{-1}\, dv \right)^2 d\omega \\
&= \int_0^\infty \int_0^\infty \left(\omega^{\frac{1}{2}} \left|\hat{F}\big(\omega, v/\omega\big)\right| / v^2 \right) dv\, d\omega +
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{0}\int_{0}^{\infty}\Big(|\omega|^{\frac{1}{2}}\left|\hat{F}\big(\omega,-v/\omega\big)\right|/\,|v|\Big)^{2}dv\,d\omega\\
&=\int_{0}^{\infty}\int_{0}^{\infty}\left|\omega^{\frac{1}{2}}\hat{F}\left(\omega,y\right)\right|^{2}y^{-2}\omega^{-1}dy\,d\omega\\
&+\int_{-\infty}^{0}\int_{-\infty}^{0}|\omega|^{\frac{1}{2}}\left|\hat{F}\left(\omega,y\right)\right|^{2}y^{-2}|\omega|^{-1}dy\,d\omega\\
&=\|F\|^{2}_{L^{2,-2}(U)},
\end{aligned}
$$

which implies

$$
L^{2,-2}\left(U\right)=\bigoplus_{\lambda\in\Lambda}P^{\psi^{\lambda}}L^{2}\left(\mathbb{R}\right)=\bigoplus_{\lambda\in\Lambda}A^{\psi^{\lambda}}.
$$

Necessity: Suppose (2.7) holds, but we have $0\neq\varphi\in L^{2}[0,\infty)$, which is orthogonal to $\omega^{-\frac{1}{2}}\psi^{\lambda}\left(\omega\right)|_{\omega\geq0}$ for any $\lambda\in\Lambda$. Then from the above proof we must have

$$
\int_{0}^{\infty}\varphi\left(v\right)\omega^{\frac{1}{2}}\hat{F}\left(\omega,v/\omega\right)v^{-1}dv=0,
$$

for any $\omega\in(0,\infty)$, $F\in L^{2,-2}\left(U\right)$. Therefore we must have $\varphi=0$, which is a contradiction, and our proof is complete. □

Thus we have given all the decomposition of $L^{2,-2}\left(U\right)$ by continuous wavelet transforms.

Note that

$$
\left\{\varphi_{n,k}\left(x\right)=2^{\frac{n}{2}}\varphi\left(2^{n}x-k\right)\right\}_{k\geq0,n\in\mathbb{Z}},\qquad\left\{\varphi_{n,k}\left(x\right)\right\}_{k>0,n\in\mathbb{Z}}
$$

are orthonormal bases of $L^{2}[0,\infty)$ and $L^{2}(-\infty,0]$ respectively, where $\varphi\left(x\right)=\chi_{[0,\frac{1}{2})}-\chi_{[\frac{1}{2},1)}$, we can decompose $L^{2,-2}\left(U\right)$ as follows.

For $F\in L^{2,-2}\left(U\right)$, we have the orthogonal decomposition

$$
F=\sum_{k\in\mathbb{Z},l\geq0}P^{k,l}\left(F\right),
$$

where $P^{k,l}\left(F\right)\left(x,y\right)$ can be determined by its Fourier transform for the first variable:

$$
\left(P^{k,l}F\right)^{\hat{}}\left(\omega,y\right)|_{\omega\geq0}=y^{\frac{1}{2}}\varphi_{k,l}\left(y\omega\right)\int_{0}^{\infty}v^{\frac{1}{2}}\sigma_{k,l}\left(v\omega\right)\hat{F}\left(\omega,v\right)v^{-2}dv,
$$

and $\left(P^{k,l}F\right)^{\hat{}}\left(\omega,y\right)|_{\omega>0}$ can be written in the same way.

In the following sections we shall give another decomposition by means of Laguerre polynomials.

3. TOEPLITZ TYPE OPERATORS

Now suppose $\{\psi^\lambda\}_{\lambda\in\Lambda} \subset AAW$ satisfies $L^{2,-2}(U) = \bigoplus_{\lambda\in\Lambda} A^{\psi^\lambda}$. We can define Toeplitz type operators $T_b^{\lambda,\mu}$ as

$$T_b^{\lambda,\mu} = P^{\psi^\lambda} M P_b \psi^\mu \tag{3.1}$$

with anti-analytic symbol $b(z)$ on U, here M_b is the operator of multiplication by b.

To characterize the Schatten-Von Neumann class S_p for the Toeplitz type operators, we need the analytic Besov spaces $B_p(U)$ on U. The space $B_p(U)$ $(0 < p < \infty)$ consists of all analytic functions on U for which the integral

$$|F|_{B_p}^p = \int_U \left|y^m F^{(m)}(z)\right|^p y^{-2} dx\, dy$$

is finite and $B_\infty(U)$ is the Bloch space, i.e., $F(z)$ analytic on U and $\|F\|_{B_\infty} = \sup_{z\in U} \left|y^m F^{(m)}(z)\right|$ is finite, here m is any integer such that $m > 1/p$.

If we compose $T_b^{\lambda,\mu}$ with the isometries T^ψ and τ^{ψ^λ}, we have $t_b^{\lambda,\mu} := \tau^{\psi^\lambda} T_b^{\lambda,\mu},\ T^{\psi^\mu} \in B\left(L^2(\mathbb{R}), L^2(\mathbb{R})\right)$, and for $f \in L^2(\mathbb{R})$, we have

(3.2)

$$\begin{aligned}
&\left(t_b^{\lambda,\mu} f\right)(\zeta) = \\
=&\int_0^\infty y^{\frac12} \hat\psi^\lambda(y\zeta)\, y^{-2} dy \left(T_b^{\lambda,\mu} T^{\psi^\eta} f\right)\hat{}\,(\zeta, y) \\
=&\int_0^\infty y^{\frac12} \hat\psi^\lambda(y\zeta)\, y^{-2} dy \int_0^\infty y^{\frac12} \overline{\hat\psi^\lambda(y\zeta)} v^{\frac12} \hat\psi^\lambda(v\zeta) \left(b(\cdot + \mathrm{i}v)\, \psi_v^\mu * f(\cdot)\right)\hat{}\,(\zeta)\, v^{-2} dv \\
=&\int_0^\infty \left|\hat\psi^\lambda(y\zeta)\right|^2 y^{-1} dy \int_0^\infty v^{\frac12} \hat\psi^\lambda(v\zeta) \int_{\mathbb{R}} \hat b(\zeta - \eta)\, e^{-\eta^{v(\eta-\zeta)}} v^{-\frac32} \psi(v\eta) \hat f(v\eta)\, d\eta dv \\
=&\int_{\mathbb{R}} \hat b(\zeta - \eta)\, \hat f(\eta)\, A(\zeta, \eta)\, d\eta,
\end{aligned}$$

where

$$A(\zeta, \eta) = \int_0^\infty \hat\psi^\lambda(v\zeta)\, \overline{\hat\psi^\mu(v\eta)} e^{-v(\eta-\zeta)} v^{-1} dv. \tag{3.3}$$

Such an operator $t_b^{\lambda,\mu}$ is called "paracommutator" [4], [9].

Now let us choose a class $\{\psi^k : k \in \mathbb{Z}_+\}$ in AAW as

$$\hat\psi^k(\zeta) = \left(\tfrac{4(k+2m+1)!}{k!}\right)^{-\frac12} \left(4|\zeta|\right)^{m+\frac12} |\zeta|^{\frac12} e^{-2|\zeta|} L_k^{(2m+1)}\left(4|\zeta|\right), \tag{3.4}$$

where $\{L_k^{(2m+1)}\}_{k\in\mathbb{Z}}$ are the Laguerre polynomials [10]

$$L_k^{(2m+1)}(x)=\sum_{s=0}^{k}\tbinom{k+2m+1}{k-s}\tfrac{(-x)^s}{s!},$$

which satisfy the orthonormal conditions:

$$\int_0^\infty e^{-x}x^{2m+1}L_n^{(2m+1)}L_k^{(2m+1)}(x)\,dx=\tfrac{(2m+1+n)!}{n!}\delta_{k,n}. \tag{3.5}$$

Then it is easy to verify that $\{\omega^{-\frac{1}{2}}\hat{\psi}^k(\omega)|_{\omega\geq 0}\}_{k\in\mathbb{Z}_+}$ $(\{|\omega|^{-\frac{1}{2}}\hat{\psi}^k(\omega)|_{\omega\leq 0}\}_{k\in\mathbb{Z}_+})$ respectively is an orthonormal basis of $L^2[0,\infty)$ $\big(L^2(-\infty,0]\big)$ respectively. Therefore we can construct the Toeplitz type operators as (3.1) and obtain the following main result.

Theorem 3.1. *Let $T_b^{k,l}$ defined as above, $k,l\in\mathbb{Z}_+$, then*

(1) *For $k=1$, we have*
   $T_b^{k,l}$ is bounded, if and only if $\bar{b}\in L^\infty$;
   $T_b^{k,l}$ is compact, if and only if $b=0$.

(2) *For $k\neq 1$, we have*
   $T_b^{k,l}\in S_p$, $1<p\leq\infty$, if and only if $\bar{b}\in B_p(U)$.

Proof. Note that $T^l\left(\tau^k\right)$ is an isometry between $A^{\hat{\psi}^l}(A^{\psi^k})$ and $L^2(\mathbb{R})$, we know that $T_b^{k,l}$ satisfies one of the above statements, if and only if $t_b^{k,l}=\tau^kT_b^{k,l}T^l$ satisfies the same statement. From (3.2), we see that $t_b^{k,l}$ is a para-commutator [4], [9] with $A^{k,l}(\zeta,\eta)$ defined by (3.3).

We can now compute $A^{k,l}(\zeta,\eta)$ as follows:

$$\begin{aligned}
&A^{k,l}(\zeta,\eta)=\\
&=\int_0^\infty \hat{\psi}^k(v\zeta)\overline{\hat{\psi}^l(v\eta)}e^{-v(\eta-\zeta)}v^{-1}dv\\
&=C_{k,m}C_{l,m}\int_0^\infty (4v|\zeta|)^{m+\frac{1}{2}}(v|\zeta|)^{\frac{1}{2}}e^{-2|v\zeta|}L_k^{(2m+1)}\big(4v|\zeta|\big)\cdot\\
&\quad\cdot\big(4v|\eta|\big)^{m+\frac{1}{2}}\big(v|\eta|\big)^{\frac{1}{2}}e^{-2v|\eta|}L_1^{(2m+1)}\big(4v|\eta|\big)e^{-v(\eta-\zeta)}v^{-1}dv\\
&=C_{k,m}C_{l,m}4^{2^{m+1}}(|\zeta\eta|)^{m+1}\cdot\\
&\quad\cdot\int_0^\infty v^{2m+2}e^{-v(2|\zeta|+2|\eta|+\eta-\eta)}L_k^{(2m+1)}\big(4v|\zeta|\big)L_k^{(2m+1)}\big(4v|\eta|\big)v^{-1}dv\\
&=C_{k,m}C_{1,m}4^{2m+1}(|\zeta\eta|)^{m+1}\sum_{s=0}^{k}\sum_{t=0}^{l}\tbinom{k+2m+1}{k-s}\tbinom{l+2m+1}{1-t}\tfrac{1}{s!t!}\cdot\\
&\quad\cdot(-1)^{s+t}\int_0^\infty\big(4v|\zeta|\big)^s\big(4v|\eta|\big)^t v^{2m+1}e^{-v(2|\zeta|-\zeta+2|\eta|+\eta)}dv
\end{aligned}$$

$$= C_{k,1}C_{l,m}4^{2m+1}\left(|\zeta\eta|\right)^{m+1}\sum_{s=0}^{k}\sum_{t=0}^{l}(-4)^{s+t}\binom{k+2m+1}{k-s}\binom{l+2m+1}{1-t}\frac{1}{s!t!}\cdot$$
$$\cdot|\zeta|^s|\eta|^t\left(2|\zeta|-\zeta+2|\eta|+\eta\right)^{-s-t-2m-2}(s+t+2m+1)!$$
$$= C_{k,m}C_{l,m}4^{2m+1}\left(|\zeta\eta|\right)^{m+1}\left(2|\zeta|-\zeta+2|\eta|+\eta\right)^{-2m-2}\cdot$$

(3.6)
$$\cdot\sum_{s=0}^{k}\sum_{t=0}^{l}(-4)^{s+t}\binom{k+2m+1}{k-s}\binom{l+2m+1}{1-t}\frac{(s+t+2m+1)!}{s!t!}\left(\frac{|\zeta|}{2|\zeta|-\zeta+2|\eta|+\eta}\right)^s\left(\frac{|\eta|}{2|\zeta|-\zeta+2|\eta|+\eta}\right)^t.$$

In the domain $\{(\zeta,\eta):\zeta,\eta>0\}$, we have

(3.7)
$$A^{k,l}(\zeta,\eta)=C_{k,m}C_{l,m}4^{2m+1}h_1^{k,l}\left(\tfrac{\zeta}{\eta}\right),$$

where for $u>0$,

(3.8)
$$h_1^{k,l}(u)=u^{m+1}(u+3)^{-2m-2}\cdot$$
$$\cdot\sum_{s=0}^{k}\sum_{t=0}^{l}(-4)^{s+t}\binom{k+2m+1}{k-s}\binom{l+2m+1}{1-t}\frac{(s+t+2m+1)!}{s!t!}\left(\frac{u}{u+3}\right)^s\left(\frac{1}{u+3}\right)^t$$
$$=(u+3)^{-2m-k-1}q_1^{k,l}(u),$$

here $q_1^{k,l}(u)$ is a polynomial with degree $\le m+1+k+1<2m+2+k+1$. Hence $h_1^{k,l}\in C^\infty[0,\infty)$, and we have

$$\left\|(h_1^{k,l})'\right\|_{L_\infty[0,\infty)}\le C_{k,l,m},$$
$$\left\|(h_1^{k,l})''\right\|_{L_\infty[0,\infty)}\le C_{k,l,m},$$

with a constant $C_{k,l,m}$ depending only on k,l, and m.

On the other hand, we know from (3.8)

$$\lim_{u\to\infty}\sum_{s=0}^{k}\sum_{t=0}^{l}(-4)^{s+t}\binom{k+2m+1}{k-s}\binom{(l+2m+1)}{1-t}\frac{(s+t+2m+1)!}{s!t!}\left(\frac{u}{u+3}\right)^s\left(\frac{1}{u+3}\right)^t=$$
$$=\sum_{s=0}^{k}(-4)^s\binom{k+2m+1}{k-s}\binom{l+2m+1}{1}\frac{(s+2m+1)!}{s!}$$
$$=\binom{l+2m+1}{1}\frac{(k+2m+1)!}{k!}(-3)^k\ne 0.$$

Thus $h_1^{k,l}$ is a nonzero rational function on $[0,\infty)$ and $A_1^{k,l}(\zeta,\eta)\ne 0$. Then we know that the assumption $A4$ in [4] is satisfied for $A_1^{k,l}$. From (3.6), we also know that $A0$ in [4] is satisfied.

The expressions of $A_1^{k,l}$ in the other three domains can be given in the same way. From these formulas, it is easy to check that A1 and A2 in [4] are satisfied for $A_1^{k,l}(\zeta,\eta)$.

As an example, we shall show $A1$ in the domain $\{(\zeta,\eta):\zeta,\eta>0\}$. In this case, for $\Delta_i=\{\zeta:\zeta\in[2^i,2^{i+1}]\}$, $\tilde{\Delta}_i=\{\zeta:\zeta\in[2^{i-1},2^{i+2}]\}$, we have from Lemma 3.9 in [4] that

$$\begin{aligned}&\|A^{k,l}\|_{M(\Delta_i\times\Delta_j)}\le\\&\le C\sup_{|\alpha|+|\beta|\le 2}\ \sup_{\zeta\in\tilde{\Delta}_i,\eta\in\tilde{\Delta}_j}\left\{|\zeta|^{|\alpha|}\,|\eta|^{|\beta|}\,D_\zeta^\alpha D_\eta^\beta A^{k,l}(\zeta,\eta)\,|\right\}\\&\le C\,|C_{k,m}C_{l,m}|\,4^{2m+1}\sup_{|\alpha+\beta|\le 2}\ \sup_{\zeta\in\tilde{\Delta}_i,\eta\in\tilde{\Delta}_j}\left\{|\zeta|^{|\alpha|}\,|\eta|^{|\beta|}\,D_\zeta^\alpha D_\eta^\beta h_1^{k,l}(\zeta/\eta)\right\}.\end{aligned}$$

The next computation is now clear:

For $(\alpha,\beta)=(\alpha,0)$, we have

$$\begin{aligned}&\sup_{\zeta\in\tilde{\Delta}_i,\eta\in\tilde{\Delta}_j}\left\{\zeta^{|\alpha|}\eta^{-\alpha}\left|(h_1^{k,l})^{(\alpha)}(\zeta/\eta)\,\right|\right\}\le\sup_{u\in(0,\infty)}\left\{\left|u^\alpha(h_1^{k,l})^{(\alpha)}(u)\,\right|\right\}\\&=\sup_{u\in(0,\infty)}\left\{\left|\,(u+3)^{-2m-2-k-1-\alpha}\,q_{2,\alpha}^{k,l}(u)\,\right|\right\}\le C'_{k,l,m}<\infty,\end{aligned}$$

where $C'_{k,l,m}$ is a constant depending only on k,l and m. Here we have used the fact that $q_{2,\alpha}^{k,1}(u)$ is a polynomial with degree $\le\alpha+m+1+k+1<2m+2+k+1+\alpha$.

For $(\alpha,\beta)=(1,1)$, we have

$$\begin{aligned}\sup_{\zeta\in\tilde{\Delta}_j,\eta\in\tilde{\Delta}_j}\left\{|\zeta|\,|\eta|\,\left|D_\zeta^1D_\eta^1h_1^{k,l}(\zeta/\eta)\right|\right\}&\le\sup_{u\in(0,\infty)}\left\{\left(\left|u(h_1^{k,l})'(u)\,\right|+\left|u^2(h_1^{k,l})''(u)\,\right|\right)\right\}\\&\le C''_{k,l,m}<\infty.\end{aligned}$$

For $(\alpha,\beta)=(0,\beta)$, we also have

$$\begin{aligned}\sup_{\zeta\in\tilde{\Delta}_i,\eta\in\tilde{\Delta}_j}\left\{|\eta|^{|\beta|}\ \left|D_\eta^\beta h_1^{k,l}(\zeta/\eta)\,\right|\right\}&\le\sup_{y\in(0,\infty)}\left\{\left(\left|u(h_1^{k,l})'(u)\,\right|+\left|u^2(h_1^{k,l})''(u)\right|\right)\right\}\\&\le C''_{k,l,m}<\infty.\end{aligned}$$

Thus we have proved that A1 in [4] is satisfied for $A^{k,l}(\zeta,\eta)$.

In the same way, from Lemma 3.8 in [4], we can prove that for $k\ne 1$ A3 (1) in [4] is satisfied as follows:

For $\zeta_0>0,\ 0<r<\zeta_0/8$, we have

$$\|A^{k,l}\|_{M(B(\zeta_0,r)\times B(\zeta_0,r))}\le C|C_{k,m}C_{l,m}|\,4^{2m+1}\sup_{|\alpha|\le 2}r^{|\alpha|}\sup_{\zeta,\eta\in B(\zeta_0,2r)}\left\{\left|D^\alpha h_1^{k,l}\left(\tfrac{\zeta}{\eta}\right)\right|\right\}.$$

Now from (3.3), we have for $\zeta=\eta>0,\ k\ne 1$

$$A^{k,l}(\zeta,\zeta)=\int_0^\infty\hat{\psi}^k(v\zeta)\,\hat{\psi}^l(v\zeta)\,v^{-1}dv=\int_0^\infty v^{-\frac{1}{2}}\hat{\psi}^k(v)\,v^{-\frac{1}{2}}\hat{\psi}(v)\,dv=0,$$

which implies $h_1^{k,l}(1)=0$ for $k\neq 1$.

Hence from $h_1^{k,l}\in C^\infty(0,\infty)$, for $\zeta,\eta\in B(\zeta_0,2r)$

$$|h_1^{k,l}(\tfrac{\zeta}{\eta})|=|h_1^{k,l}(\tfrac{\zeta}{\eta})-h_1^{k,l}(\tfrac{\eta}{\eta})|\le|\tfrac{\zeta-\eta}{\eta}|\cdot\|(h_1^{k,l})'\|_{L_\infty[\frac12,2]}\le 8\|(h_1^{k,l})'\|_{L_\infty[\frac12,2]}\cdot\tfrac{r}{\zeta_0}.$$

For $|\alpha|\ge 1$, we have

$$\sup_{\zeta,\eta\in B(\zeta_0,2r)}\left\{r^{|\alpha|}|D^\alpha h_1^{k,l}(\tfrac{\zeta}{\eta})|\right\}\le 40r_0\sum_{0\le i,j\le 2}\|u^i(h_1^{k,l}(u))^{(j)}\|_{L_\infty[\frac12,2]}\cdot\zeta_0.$$

Thus we have proved A3 (1) for $\zeta_0>0$. For $\zeta_0<0$, the proof is almost the same, and we shall omit it.

Our proof for (2) is now complete by the result in [4].

To prove (1), we need to check A8 in [4]. From (3.3), we have for $\zeta=\eta>0,\ k=1$

$$A^{k,1}(\zeta,\eta)\int_0^\infty\hat\psi^k(v\zeta)\,\overline{\hat\psi^k(v\zeta)}v^{-1}dv=\int_0^\infty|\hat\psi^k(\omega)|^2\omega^{-1}d\omega=1.$$

On the other hand, note that $A^{k,l}\in C^\infty\left([\frac12,2]\times[\frac12,2]\right)$, we have A8 by [4].

Thus our proof of (1) is also complete since A_1,A_2, A8 are satisfied. □

Hankel type operators can also be discussed as in [5], [6], and we have similar results.

REFERENCES

[1] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inf. Theory, **36** (1990), 961–1005.

[2] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure and Appl. Math., **41** (1983), 909–996.

[3] A. Grossmann, J. Morlet and T. Paul, *Transforms associated to square integrable group representation II: examples*, Ann. Inst. Henri Poincaré, **45** (1986), 293–309.

[4] S. Janson and J. Peetre, *Paracommutator-boundedness and Schatten-operties*, Trans. Amer. Math. Soc., **305** (1988), 467–504.

[5] Q. Jiang and L. Peng; *Wavelet transform and Hankel-Toeplitz operators*; Preprint.

[6] Q. Jiang and L. Peng, *Toeplitz and Hankel type operators on the upper half-plane*, Preprint.

[7] Y. Meyer, *Ondelettes et opérateurs*, Hermann (1990).

[8] T. Paul, *Functions analytic on the half-plane as quantum mechanical states*, J. Math. Phys. **25** (1985), 3252–3263.

[9] L. Peng, *Paracommutator of Schatten-Von Neumann class $S_p, 0<p<1$*, Math. Scand., **61** (1987), 68–92.

[10] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publications, Vol. **23** (1939).

Received 10.XI.1991

Department of Mathematics,
Zhejiang University
Advanced Institute of Mathematics,
Hangzhou, Zhejiang
P.R. China, 310027