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SOME REMARKS CONCERNING THE MORSE-SMALE CHARACTERISTIC OF A COMPACT MANIFOLD

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1. Introduction. Let M^m be a compact m-dimensional smooth manifold without boundary ($\partial M = \emptyset$) and let $\mathscr{F}(M)$ be the real algebra of all smooth mappings $f : M \to \mathbb{R}$. For $f \in \mathscr{F}(M)$ let us define the *critical set* of f by $C[f] = \{p \in M : (df)_p = 0\}$ and the *bifurcation set* by $B[f] = f(C[f])$. The elements $p \in C[f]$ are the critical points of f and the elements of the set $B[f]$ represent the critical values of the mapping f. Recall that a critical point $p \in C[f]$ is *non-degenerate* if the bilinear form $(d^2f)_p : T_p(M) \times T_p(M) \to \mathbb{R}$ is non-degenerate, i. e. there exists a chart (U, φ) in p such that the Hessian matrix $H(f_\varphi)(\varphi(p)) = (\partial^2 f_\varphi/\partial x^i \partial x^j(\varphi(p)))_{1 \leq i,j \leq m}$ is invertible, where $f_\varphi = f \circ \varphi^{-1}$. Notice that this notion does not depends on the chart (U, φ). Taking into account the well-known Morse lemma (see [4], [6, p. 199]), it results that for a non-degenerate critical point $p \in C[f]$ there exists a chart (U, φ) with $f_\varphi(x) = f(p) - \|P^-(x)\|^2 + \|P^+(x)\|^2$ for any point $p \in \varphi(U) \subseteq \mathbb{R}^m$, where P^- and P^+ are projections on some subspaces of $\mathbb{R}^m$. From this result one obtains the following decomposition of the tangent space $T_p(M)$, $T_p(M) = T_p^-(M) \oplus T_p^+(M)$, where $T_p^-(M)$, is the maximal subspaces on which the quadratic form $(d^2f)_p(X_p, X_p)$ is negatively defined. The number $k(p) = \dim_{\mathbb{R}} T_p^-(M) = \dim_{\mathbb{R}} \mathrm{Im} P^-$ is called the *Morse index* of $p \in C[f]$. It is clear that $0 \leq k(p) \leq m$ and from the proof of the Morse lemma it results that $k(p)$ represents the number of the negative eingevalues of the Hessian matrix $H(f_\varphi)(\varphi(p))$. It is not difficult to show that the non-degenerate critical points of f are isolated in the critical set $C[f]$.

1.1. Definition. If the set $C[f]$ contains only non-degenerate critical points, the mapping $f \in \mathscr{F}(M)$ is called a *Morse function* on M.

Let us denote by $\mathscr{F}_m(M) \subset \mathscr{F}(M)$ the set of all Morse functions defined on the manifold M. The following result is a basic tool in Differential Topology.

1.2. Theorem. *For any finite-dimensional compact manifold M the relation $\mathscr{F}_m(M) \neq \emptyset$ holds, i. e. there exists a Morse function defined on M.*

For details concerning this result we refer to the excelent book [4].

If $f \in \mathscr{F}_m(M)$, the critical set $C[f]$ is finite because it contains only isolated critical points and the manifold M is compact. Denote by $\mu_k(f)$ the number of the critical points of f with the Morse index k, $0 \leq k \leq m$.

Consider $H_k(M; F)$, $k = \overline{0, m}$, the singular homology groups with the coefficients in the field F and $\beta_k(M; F) = \text{rank } H_k(M; F) = \dim_F H_k(M; F)$, $k = \overline{0, m}$, the Betti numbers of the manifold M.

1.3. **Theorem.** *If $f \in \mathcal{F}_m(M)$ the following relations hold*

$\mu_k(f) \geqslant \beta_k(M; F)$, $k = \overline{0, m}$ *(weak Morse inequalities)*

$$\sum_{k=0}^{m} (-1)^k \mu_k(f) = \chi(M) \text{ (Euler formula)}$$

For proof and applications of these very important relations we refer to the book of Palais, R. S., Terng, Chun-lian [6, p 213–222] or Andrica, D. [1, p 71–80].

1.4. **Definition.** A Morse function $f \in \mathcal{F}_m(M)$ is called *F-perfect* if $\mu_k(f) = \beta_k(M; F)$, $k = \overline{0, m}$.

The existence of an F — perfect Morse function on the manifold M is an important problem with many topological and geometrical consequences.

2. The Morse-Smale characteristic and the numbers $\gamma_i(M)$

If $f \in \mathcal{F}_m(M)$, let $\mu(f)$ be the integer number defined by

$$\mu(f) = \sum_{k=0}^{m} \mu_k(f) \tag{1}$$

It is obvious to see that $\mu(f)$ represents the total number of critical points of f, i. e. the cardinal number of the set $C[f]$.

2.1. **Definition.** The number

$$\gamma(M) = \min \{\mu(f) : f \in \mathcal{F}_m(M)\} \tag{2}$$

is called the *Morse-Smale characteristic* of M.

In the paper [2] it is proved that the Morse-Smale characteristic is a differential invariant of M, i. e. if the compact manifolds M, N are diffeomorphic, then $\gamma(M) = \gamma(N)$.

It is known [2, Theorem 3. 1.] that the compact manifold M has F-perfect Morse functions if and only if

$$\gamma(M) = \beta(M; F) \tag{3}$$

where $\beta(M; F) = \sum_{k=0}^{m} \beta_k(M; F)$.

In an analogous way with (2) let us define the numbers $\gamma_i(M)$, $i = \overline{0, m}$, by

$$\gamma_i(M) = \min \{\mu_i(f) : f \in \mathcal{F}_m(M)\} \tag{4}$$

It is easy to see that the following simple inequality is true

$$\gamma(M) \geqslant \sum_{i=0}^{m} \gamma_i(M) \tag{5}$$

2.2. **Theorem** 1) *The number $\gamma_i(M)$ is a differential invariant of the manifold M, $i = \overline{0, m}$.*

2) $\gamma_i(M) = \gamma_{m-i}(M)$, $i = \overline{0, m}$.

Proof. 1) Let us consider the function $\Theta : \mathcal{F}(M) \to \mathcal{F}(N)$ defined by $\Theta(f) = f \circ \varphi$, where $\varphi : N \to M$ is a difeomorphism between the manifolds N, M. Using the result of [2, Lemma 2. 3.] it follows that $\Theta \mid \mathcal{F}_m(M) : \mathcal{F}_m(M) \to \mathcal{F}_m(N)$ and $\mu_i(f) = \mu_i(f \circ \varphi)$, $i = \overline{0, m}$, for any Morse function $f \in \mathcal{F}_m(M)$. It is easy to verify that $\Theta \mid \mathcal{F}_m(M)$ is one to one and $(\Theta \mid \mathcal{F}_m(M))^{-1} = \psi \mid \mathcal{F}_m(M)$, where $\psi : \mathcal{F}(N) \to \mathcal{F}(M)$ is given by $\psi(g) = g \circ \varphi^{-1}$ i. e. the function $\Theta \mid \mathcal{F}_m(M)$ is a natural bijection between $\mathcal{F}_m(M)$ and $\mathcal{F}_m(N)$.

Using these properties and the results contained in [2, Lemma 2. 2 and Lemma 2.3.] one obtains $\gamma_i(M) = \min \{\mu_i(f) : f \in \mathcal{F}_m(M)\} = \min\{\mu_i(f \circ \varphi) : f \in \mathcal{F}_m(M)\} = \min \{\mu_i(\Theta(f)) : f \in \mathcal{F}_m(M)\} = \min \{\mu_i(g) : g \in \mathcal{F}_m(N)\} = \gamma_i(N)$

2) For a Morse function $f \in \mathcal{F}_m(M)$ the mapping $h = -f$ satisfies the relations $h \in \mathcal{F}_m(M)$ and $\mu_i(f) = \mu_{m-i}(h)$, $i = \overline{0, m}$. Therefore, for any Morse function $f \in \mathcal{F}_m(M)$, $\mu_i(f) \geqslant \gamma_{m-i}(M)$ and consequently $\gamma_i(M) \geqslant \gamma_{m-i}(M)$, $i = \overline{0, m}$. Replacing i with $m - i$, from the above inequalities it results $\gamma_{m-i}(M) \geqslant \gamma_i(M)$, i. e. the desired relations hold.

3. The numbers γ, γ_i on the product manifolds.

Let M^m, N^n be two compact manifolds without boundary ($\partial M = \partial N = \emptyset$).

3.1. **Proposition.** The following relations hold:

1) $\gamma(M \times N) \leqslant \gamma(M)\, \gamma(N)$

2) $\gamma_i(M \times N) \leqslant \sum_{j+k=i} \gamma_j(M)\, \gamma_k(N)$, $i = \overline{0, m+n}$

Proof. 1) Consider $f \in \mathcal{F}_m(M)$, $g \in \mathcal{F}_m(N)$ and $h : M \times N \to \mathbb{R}$ given by $h(x, y) = f(x) + g(y)$. It is easy to see that $C[h] = C[f] \times C[g]$, thus $\mu(h) = \mu(f)\mu(g)$. On the other hand, after an elementary calculus, it results that the Hessian matrix of h in (p, q) is

$$H(h)(p, q) = \left(\begin{array}{c|c} H(f)(p) & 0 \\ \hline 0 & H(g)(q) \end{array} \right),$$

i. e. h is a Morse function on $M \times N$. Taking into account Definition 2.1 and the relation $\mu(h) = \mu(f)\, \mu(g)$, one obtains $\gamma(M \times N) \leqslant \mu(f)\, \mu(g)$ for any Morse functions $f \in \mathcal{F}_m(M)$ and $g \in \mathcal{F}_m(N)$. That is $\gamma(M \times N) \leqslant \gamma(M)\gamma(N)$.

2) With the above notation let us remark that $\mu_i(h) = \sum_{j+k=i} \mu_j(f)\mu_k(g)$, for any Morse functions $f \in \mathcal{F}_m(M)$, $g \in \mathcal{F}_m(N)$. According to the definition of the number γ_i (see relation (4)), the desired relations follow.

It is a natural and important problem to get the manifolds M, N which satisfy the equalities in Proposition 3.1 (see Kuiper, N. H. [5] and Rassias, G. M. [9]). A sufficient condition is given in the following result.

3.2. Theorem. *If M^m, N^n are compact manifolds without boundary ($\partial M = \partial N = \emptyset$) which have F-perfect Morse functions, then* $\gamma(M\times N) = \gamma(M)\gamma(N)$ *and* $\gamma_i(M\times N) = \sum_{j+k=i}\gamma_j(M)\gamma_k(N)$, $i = \overline{0, m+n}$.

Proof. Taking into account relation (3) it results $\gamma(M) = \beta(M; F)$ and $\gamma(N) = \beta(N; F)$. Consider $P(t; M, F) = \sum_{p=0}^{m}\beta_p(M; F)t^p$, $P(t; N, F) = \sum_{q=0}^{n}\beta_q(N; F)t^q$, the Poincaré polynomials of the manifolds M, N. It results $P(t; M, F)\, P(t; N, F) = \sum_{k=0}^{m+n}(\sum_{p+q=k}\beta_p(M; F)\,\beta_q(N; F))t^k$. Because the homology coefficients group is the field F, from the well-known Kunneth formula, one obtains

$$H_k(M\times N; F) \simeq \bigoplus_{p+q=k}(H_p(M; F)\otimes H_q(N; F)),\ k = \overline{0, m+n}, \quad (6)$$

(see for example [3, p 108]). Taking into account (6) the following relations $\beta_k(M\times N; F) = \sum_{p+q=k}\beta_p(M; F)\beta_q(N; F)$, $k = \overline{0, m+n}$, hold i. e. $P(t; M, F)P(t; N, F) = P(t; M\times N, F)$. On the other hand it is not dificult to show that the manifold $M\times N$ has F-perfect Morse functions (for example, if $f\in\mathcal{F}_m(M)$, $g\in\mathcal{F}_m(N)$ are F-perfect, then $h\in\mathcal{F}_m(M\times N)$, $h(x, y) = f(x) + g(y)$ is F-perfect too; see the proof of Proposition 3.1). Using the equality $P(1; M\times N, F) = P(1; M, F)\,P(1; N, F)$ and applying relation (3) for M, N and $M\times N$ it follows $\beta(M\times N; F) = \beta(M; F)\beta(N; F)$, i. e. $\gamma(M\times N) = \gamma(M)\gamma(N)$.

The second equality follows in an analogous way.

3.3. Corollary. *Let M^m, N^n be compact manifolds without boundary ($\partial M = \partial N = \emptyset$), $m, n \geqslant 6$. If the singular homology groups $H_k(M; \mathbb{Z})$, $H_j(N; \mathbb{Z})$ are torsion-free, then* $\gamma(M\times N) = \gamma(M)\gamma(N)$ *and* $\gamma_i(M\times N) = \sum_{j+k=i}\gamma_j(M)\gamma_k(N)$, $i = \overline{0, m+n}$.

Proof. From [2, Corollary 3, 4.] one obtains that M, N have Q-perfect Morse functions. Using Theorem 3.2, the desired result follows.

3.4. Remarks 1) Taking into account the equality $\gamma(S^m) = 2$ (see [2, Example 3. 6]), it follows $\gamma(S^{m_1}\times\ldots\times S^{m_k}) = 2^k$. For example, if $T^k = S^1\times\ldots\times S^1$ is the k-dimensional torus, then $\gamma(T^k) = 2^k$. Using, the second equality contained in Theorem 3.2, one obtains $\gamma_i(T^k) = C_k^i$, $i = \overline{0, m}$.

2) Under the hypotheses of Theorem 3.2, inequality (5) becomes an equality, i. e. a partial answer for another problem of Rassias, G.M. [7], [8] (see also Andrica, D. [2]) is obtained.

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