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SOME REMARKS ON CONVEX FUNCTIONS

SEVER S. DRAGOMIR and NICOLETA M. IONESCU
(Timişoara)

1. Let $f : C \subseteq X \to \mathbb{R}$ be a convex mapping on convex subset C of linear space X. For a, b two given elements in C, we shall define the following mapping of real variable $F(a, b) : [0, 1] \to \mathbb{R}$ given by :

$$F(a, b)(t) := \frac{1}{2}\, [f(ta + (1-t)\, b) + f((1-t)\, a + tb)]$$

which is well-defined for all t in $[0, 1]$.

The next theorem contains some remarkable properties of this mapping.

Theorem. *In the above assumptions, we have* ;

(i) $F(a,b)\,(\tau + 1/2) = F\,(a, b)\,(1/2 - \tau)$ *for all* τ *in* $[0, 1/2]$;

(ii) $\sup_{t\in[0,1]} F(a, b)\,(t) = F\,(a, b)\,(0) = F(a, b)\,(1) = \dfrac{f(a) + f(b)}{2}$;

(iii) $\inf_{t\in[0,1]} F(a, b)(t) = F(a, b)\,(1/2) = f\left(\dfrac{a+b}{2}\right)$;

(iv) $F(a, b)$ *is convex on* $[0, 1]$;

(v) *We have the generalised Hadamard's inequalities* [6] :

$$f\left(\frac{a+b}{2}\right) \leqslant \int_0^1 f(ta + (1-t)\, b)\; dt \leqslant \frac{f(a) + f(b)}{2}; \tag{1}$$

(vi) *Let* $p_i \geqslant 0$ *with* $P_n := \sum_{i=1}^{n} p_i > 0$ *and* t_i *are in* $[0, 1]$ *for all* $i = 1, \ldots, b$. *Then we have the inequality* :

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant F\,(a, b)\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i t_i\right) \leqslant \\ &\leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i F(a, b)\,(t_i) \leqslant \frac{f(a) + f(b)}{2} \end{aligned} \tag{2}$$

which is a discrete variant of Hadamard's result;

Moreover, if we assume that $X = \mathbb{R}$ *and* $a, b \in C$, $a < b$, *here* C *is an interval of real numbers, we also have* :

(vii) $F\,(a, b)$ *is monotonous decreasing on* $[0, 1/2]$ *and monotonous increasing on* $[1/2, 1]$;

(viii) *We have the identity* :

$$\int_0^1 F(a,b)(t)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx\,;$$

(ix) *The Hadamard's inequalities hold, e. e.*,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}\,;$$

(x) *If f is differentiable on* $[a, b]$, *then* :

$$F(a,b)(t) \geqslant \max\left\{f(a) + \frac{1}{2}(b-a)f'(a),\ f(b) - \frac{1}{2}(b-a)f'(b)\right\}$$

for all t in $[0, 1]$;

(xi) *If f is as in* (*x*), *we also have* :

$$f\left(\frac{a+b}{2}\right) \geqslant \frac{f(a)+f(b)}{2} - \frac{b-a}{2}\,(f'(b) - f'(a)).$$

Proof. (i) A simple computation shows that :

$$F(a,b)(\tau + 1/2) = \frac{1}{2}\,[f((\tau + 1/2)\,a + (1/2 - \tau)\,b) +$$

$$+ f((1/2 - \tau)\,a + (\tau + 1/2)\,b] = F(a,b)(1/2 - \tau),$$

for all τ in $[0, 1/2]$, which proves the statement.

(ii) Using the convexity of f we get :

$$F(a,b)(t) \leqslant \frac{1}{2}\,[tf(a) + (1-t)f(b) + (1-t)\,f(a) + tf(b)] = \frac{f(a)+f(b)}{2}$$

for all t in $[0, 1]$ and

$$F(a,b)(0) = F(a,b)(1) = \frac{f(a)+f(b)}{2},$$

which proves the assertion.

(iii) By the convexity of f we also have :

$$F(a,b)(t) \geqslant f\left[\frac{ta + (1-t)\,b + (1-t)\,a + tb}{2}\right] = f\left(\frac{a+b}{2}\right)$$

for all t in $[0, 1]$ and

$$F(a,b)(1/2) = f\left(\frac{a+b}{2}\right)$$

which shows the statement.

(iv) Let $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$ and t_1, t_2 are in $[0, 1]$. Then

$$F(a,b)(\alpha t_1 + \beta t_2) = \frac{1}{2}\,[f(\alpha[t_1 a + (1-t_1)\,b] + \beta[t_2 a + (1-t_2 b)] +$$

$$+ f(\alpha[(1-t_1)\,a + t_1 b] + \beta\,[(1-t_2)\,a + t_2 b]) \leqslant \frac{1}{2}\,[\alpha f(t_1 a + (1-t_1)\,b) +$$

$$+ \beta f(t_2 a + (1-t_2)\,b) + \alpha f((1-t_1)\,a + t_1 b) + \beta f((1-t_2)\,a + t_2 b)] =$$

$$= \alpha F(a,b)(t_1) + \beta F(a,b)(t_2),$$

which shows that $F(a, b)$ is convex on $[0, 1]$.

(v) $F(a,b)$ being convex on $[0, 1]$, it is integrable on $[0, 1]$ and by (ii) and (iii) we get :

$$f\left(\frac{a+b}{2}\right) \leqslant \int_0^1 F(a,b)(t)\,dt \leqslant \frac{f(a)+f(b)}{2}.$$

Since a simple calculus shows that

$$\int_0^1 F(a,b)(t)\,dt = \int_0^1 f(ta + (1-t)\,b)\,dt,$$

the proof of inequality (1) is finished.

(vi) The first inequality in (2) is obvious from (iii).

The second inequality follows by Jensen's inequality applied for the convex mapping $F(a, b)$.

To prove the last inequality in (2), we observe, by (ii), that :

$$F(a,b)(t_i) \leqslant \frac{f(a)+f(b)}{2}$$

for all $i = 1, \ldots, n$. By multiplying with $p_i \geqslant 0$ and summing these inequalities after i to 1 at n, we obtain the desired inequality.

(vii) $F(a, b)$ being convex on (0,1), for all $t_2 > t_1$, with $t_1, t_2 \in [1/2, 1)$, we have :

$$(F(a,b)(t_2) - F(a,b)(t_1))/(t_2 - t_1) \geqslant F'(a,b)(t_1) =$$

$$= \frac{b-a}{2}\,[f'_+((1-t_1)\,a + t_1 b) - f'_+(t_1 a + (1-t_1)\,b)].$$

Since $t_1 \in [1/2, 1)$, we have $(1-t_1)a + t_1 b \geqslant t_1 a + (1-t_1)b$ and because f'_+ is monotonous increasing on (a, b), we deduce that :

$$f'_+((1-t_1)\,a + t_1 b) \geqslant f'_+(t_1 a + (1-t_1)\,b),$$

i. e., $F(a, b)$ is monotonous increasing on $[1/2, 1)$ and, by (ii), also in $[1/2, 1]$.

The fact that $F(a,b)$ is monotonous decreasing on $[0, 1/2]$ goes likewise and we omit it.

(viii) It's obvious observing that

$$\int_0^1 f(ta + (1-t)\,b)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(ix) Follows by (v) and (viii).

(x) Since f is differentiable on $[a, b]$, we have:

$$f(ta + (1-t)\,b) \geq f(a) + (1-t)\,(b-a)\,f'(a)$$

$$f((1-t)\,a + tb) \geq f(a) + t(b-a)\,f'(a),$$

for all t in $[0, 1]$. Summing these inequalities, we get:

$$F(a, b)\,(t) \geq f(a) + \frac{b-a}{2}\,f'(a).$$

The fact that:

$$F(a, b)\,(t) \geq f(b) - \frac{b-a}{2}\,f'(b),\; t \in [0, 1],$$

goes likewise and we omit the details.

(xi) It's obvious from (iii) and (x). We omit the details.

For other inequalities for convex functions see [3–6] where further references are given.

2. Applications. 1. Let $(X, \|\cdot\|)$ be a normed linear space and x, y be two given elements in X. Then for all $p \geq 1$ we have:

$$\left\|\frac{x+y}{2}\right\|^p \leq \frac{1}{2}\,[\|tx + (1-t)\,y\|^p + \|(1-t)\,x + ty\|^p] \leq \frac{\|x\|^p + \|y\|^p}{2}$$

for every t in $[0, 1]$ and

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|tx + (1-t)\,y\|^p\,dt \leq \frac{\|x\|^p + \|y\|^p}{2}.$$

Let $p_i \geq 0$ with $P_n > 0$ and $t_i \in [0,1]$, $i = 1, \ldots, n$. Then we have the inequality:

$$\left\|\frac{x+y}{2}\right\|^p \leq \frac{1}{2}\left[\left\|\left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)x + \left(1 - \frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)y\right\|^p + \right.$$

$$\left. + \left\|\left(1 - \frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)x + \left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)y\right\|^p\right] \leq \frac{1}{2P_n}\sum_{i=1}^n p_i [\|t_i x + (1-t_i)\,y\|^p +$$

$$+ \|(1-t_i)\,x + t_i y\|^p] \leq \frac{\|x\|^p + \|y\|^p}{2}$$

for all x, y in X.

If we assume that x, y are positive real numbers and $p \geq 1$, then we also have:

$$\frac{1}{2}\,[(tx + (1-t)\,y)^p + ((1-t)\,x + ty)^p] \geq$$

$$\geq \max\left\{x^p + \frac{p}{2}\,(y-x)\,x^{p-1},\; y^p - \frac{p}{2}\,(y-x)\,y^{p-1}\right\}$$

for all t in $[0,1]$.

The proofs follow from the above theorem for the convex mapping $f : X \to \mathbb{R}$, $f(x) = \|x\|^p$ respectively $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^p$ $(p \geq 1)$.

2. Let $0 \leq a \leq b$. Then we have the inequalities:

$$\frac{a+b}{2} \geq [(ta + (1-t)\,b)\,((1-t)\,a + tb)]^{1/2} \geq \sqrt{ab}$$

for all t in $[0, 1]$ and

$$\frac{a+b}{2} \geq \exp\int_0^1 \ln\,(ta + (1-t)\,b)\,dt \geq \sqrt{ab} \text{ (here } 0 < a \leq b).$$

Now, let $p_i \geq 0$ with $P_n > 0$ and $t_i \in [0, 1]$, $i = 1, \ldots, n$. Then one has the inequalities:

$$\frac{a+b}{2} \geq \left\{\left[\left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)a + \left(1 - \frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)b\right]\cdot\right.$$

$$\left.\cdot\left[\left(1 - \frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)a + \left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)b\right]\right\}^{1/2} \geq$$

$$\geq \left\{\prod_{i=1}^n [(t_i a + (1-t_i)\,b)\,((1-t_i)\,a + t_i b)]^{p_i/2}\right\}^{1/P_n} \geq \sqrt{ab}.$$

Finally, we also have:

$$\ln[(ta + (1-t)\,b)\,((1-t)\,a + tb)]^{1/2} \leq \min\left\{\ln a + \frac{b-a}{2a},\; \ln b - \frac{b-a}{2b}\right\}$$

for all $t \in [0, 1]$ and $0 < a \leq b$.

The proofs follow from the above theorem for the convex mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$.

Remark. If we choose $f : (0, 1/2] \to \mathbb{R}$, $f(x) = -\ln\,(x/(1-x))$ or $f : (0,1) \to \mathbb{R}$, $f(x) = \ln\,(x/(1-x))^x$ we may obtain some interesting refinements of Ky Fan's [2, p. 5] and Horst Alzer's [1] results considered in the particular case $n = 2$.

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Department of Mathematics
University of Timişoara
B-dul B. Pârvan, 4
R–1900 Timişoara
România