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ON A SCHUR-TYPE INEQUALITY IN L^2

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Abstract. This note is a first attempt to give an admissible numerical value for the constant C in the estimate $\|P\| \leqslant C\, n\, \|xP\|$ where P is an algebraic polynominal of degree at most n $(n > 0)$ and $\|\cdot\|$ denotes the L^2-metric on $[-1, 1]$.

1. Notation and result

Let f be a measurable function defined on $[-1, 1]$.

We set $\|f\|_2 = \left[\int_{-1}^{1} |f(x)|^2\, dx\right]^{1/2}$,

$$\|f\|_\infty = \operatorname{ess}\sup_{x\in[-1,1]} |f(x)|.$$

We denote by $\mathbb{P}_n$ the set of algebraic polynomials of degree at most n.

In 1919, Schur [4] gave the following estimate

$$\text{for any } P \in \mathbb{P}_n,\ \|P\|_\infty \leqslant (n+1)\, \|xP\|_\infty.$$

It is also well known (see for example [2] or [3 §6.3]) that there exists an absolute constant C such that for any $P \in \mathbb{P}_n$ $(n > 0)$

$$\|P\|_2 \leqslant Cn\ \|xP\|_2. \tag{1}$$

Furthermore exponents 1 of n in (1) is optimal.

(More generally, analogous estimates hold in the L^p-metric). This type of inequality is extensively used in approximation theory. Unfortunately a numerical value for the constant C has never been provided. So our aim is to prove the following.

Proposition. *For any* $P \in \mathbb{P}_n$ $(n \geqslant 0)$

$$\|P\|_2 \leqslant 1.13\ (n+2)\ \|xP\|_2.$$

2. Proof of the proposition.

We recall that the classical Legendre polynomials (P_i) satisfy

$$iP_i = (2i-1)\, xP_{i-1} - (i-1)\, P_{i-2}\ (i \geqslant 2)$$

$$\|P_i\|_\infty = (i + (1/2))^{-1/2}\ (i \geqslant 0)$$

Then if we denote by (p_i) the associated set of orthonormal Legendre polynomials we have

$$p_i = \left(4 - \frac{1}{i^2}\right)^{1/2} x p_{i-1} - \left(1 + \frac{1}{2i^3 - 3i^2}\right)^{1/2} p_{i-2} \quad (i \geq 2). \qquad (2)$$

Let us define a linear mapping T_n from $\mathbb{P}_n$ into $\mathbb{P}_{n-1}$ $(n > 0)$ by

$$T_n : \alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_1 x + \alpha_0 \to \alpha_n x^{n-1} + \dots + \alpha_2 x + \alpha_1$$

that is $T_n : P \to Q$, where Q is the quotient in the euclidian division of P by x. Using (2) repeatedly shows that entries a_{ij} of the matrix $A = (a_{ij})_{1 \leq i \leq n,\, 0 \leq j \leq n}$ of the operator T_n with respect to the orthonormal basis (p_i) can be computed by the following formulas using auxiliary constants b_{ij}

$$b_{ij} = 0 \text{ if } i > j \text{ or } i + j \text{ odd},$$

$$b_{jj} = 1 \quad (j = 1, 2, \dots, n)$$

$$b_{i-2,j} = -b_{ij}\left(1 + \frac{1}{2i^3 - 3i^2}\right)^{1/2} \qquad (3)$$

$$(j = 1, \dots, n;\ i = j, j-2, j-4, \dots)$$

$$a_{ij} = b_{ij}\left(4 - \frac{1}{i^2}\right)^{1/2}. \qquad (4)$$

$$(i = 1, \dots, n;\ j = 0, \dots, n)$$

Example: For $n = 3$, $A = \begin{pmatrix} 0 & \sqrt{3} & 0 & -2\sqrt{7}/3 \\ 0 & 0 & \sqrt{15}/2 & 0 \\ 0 & 0 & 0 & \sqrt{35}/3 \end{pmatrix}$.

We recall (see [1]) that

$$\| T_n \| = \text{Max } \{\| T_n P \|_2 / \| P \|_2 ;\ P \in \mathbb{P}_n,\ P \not\equiv 0\}$$

is the square root of the largest eigenvalue of the matrix ${}^tA.A$ or ${}^tB.B$ where $B = (a_{ij})_{1 \leq i \leq n,\, 1 \leq j \leq n}$ (since the first column of A contains only 0-entries).

Example. With previous matrix A $(n = 3)$ we get

$$\| T_1 \| = \sqrt{3},\ \| T_2 \| = \sqrt{15}/2,\ \| T_3 \| = (5 + \sqrt{40/3})^{1/2}.$$

In order to give an estimation of the largest eigenvalue of ${}^tB.B$ we need an evaluation of entries a_{ij} of B. Using inequality (3) we get for any i and j

$$| b_{ij} | \leq \prod_{p=1}^{\infty} \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2} \quad (j \text{ even})$$

$$| b_{ij} | \leq \prod_{p=1}^{\infty} \left(1 + \frac{1}{(2p+1)^2(4p-1)}\right)^{1/2}$$

$$\leq \prod_{p=1}^{\infty} \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2} \quad (j \text{ odd}).$$

We have

$$\prod_{p=N+1}^{\infty} \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2} \leq \prod_{p=N+1}^{\infty} \left(1 + \frac{1}{8p^2(4p-3)}\right)$$

$$= \exp\left(\sum_{N+1}^{\infty} \text{Ln}\left(1 + \frac{1}{8p^2(4p-3)}\right)\right) \leq \exp\left(\sum_{N+1}^{\infty} \frac{1}{8p^2(4p-3)}\right)$$

$$\leq \exp\left(\int_N^{\infty} \frac{dx}{8x^2(4x-3)}\right) = \left(\frac{4N}{4N-3}\right)^{1/18} \exp(-1/(24N)).$$

Then for any i, j and N we have

$$| b_{ij} | \leq \left(\frac{4N}{4N-3}\right)^{1/18} \exp(-1/(24N)) \prod_{p=1}^{N} \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2}.$$

Taking $N = 10$ yields $| b_{ij} | < 1.13$. It follows from (4) that for any i and j we have $| a_{ij} | < 2.26$. So the matrix B has the following form

$$B = \begin{pmatrix} x & 0 & x & 0 & x & \dots \\ 0 & x & 0 & x & 0 & \dots \\ 0 & 0 & x & 0 & x & \dots \\ 0 & 0 & 0 & x & 0 & \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}$$

where x denotes a non zero term whose absolute value is less than 2.26.

For $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ we set $\|v\| = (v_1^2 + v_2^2 + \ldots + v_n^2)^{1/2}$ and $Bv = ((Bv)_1, (Bv)_2, \ldots, (Bv)_n)$. We have

$$|(Bv)_1| \leqslant 2.26 \,\|(1, 0, 1, 0, \ldots)\| \, \|v\|$$

$$|(Bv)_2| \leqslant 2.26 \,\|(0, 1, 0, 1, \ldots)\| \, \|v\|$$

$$|(Bv)_3| \leqslant 2.26 \,\|(0, 0, 1, 0, \ldots)\| \, \|v\|$$

and so on.

If n is even then

$$\|Bv\| \leqslant 2.26 \,(1 + 1 + 2 + 2 + \ldots + (n/2) + (n/2))^{1/2} \|v\|$$

$$= 2.26 \,((n/2)\,((n/2) + 1))^{1/2} \|v\| \leqslant 1.13 \,(n + 1) \,\|v\|$$

and if n is odd

$$\|Bv\| \leqslant 2.26 \,(1 + 1 + 2 + 2 + \ldots + ((n-1)/2) + ((n-1)/2) + $$
$$+ ((n+1)/2)^{1/2} \|v\| = 1.13 \,(n + 1) \,\|v\|$$

so in both cases for any $v \in \mathbb{R}^n$, $\|Bv\| \leqslant 1.13 \,(n + 1) \|v\|$ and by a similar calculation $\|{}^tBv\| \leqslant 1.13 \,(n + 1) \,\|v\|$ whence

$$\|{}^tB . Bv\| \leqslant 1.13^2 \,(n + 1)^2 \,\|v\|.$$

The last estimate shows that the largest eigenvalue of ${}^tB.B$ is less than $1.13^2 \,(n + 1)^2$ and then $\|T_n\| \leqslant 1.13 \,(n + 1)$. In particular for any polynomial $xP\,(x)$ with $P \in \mathbb{P}_n (n \geqslant 0)$ we have

$$\|P\|_2 = \|T_{n+1}\,(xP)\|_2 \leqslant \|T_{n+1}\| \, \|xP\|_2 \leqslant 1.13 \,(n + 2) \, \|xP\|_2$$

which completes the proof.

In Table 1 we give numerical values of $\|T_{n+1}\|/(n + 2)$ for n in the range $[0, 32]$. The calculation was made by Dörfler's method [1] (computation of the square root of the largest eigenvalue of ${}^tA.A$) with a computer. This table shows that the coefficient 1.13 is not optimal!

Table 1

n	$\|T_{n+1}\|/(n + 2)$	n	$\|T_{n+1}\|/(n + 2)$	n	$\|T_{n+1}\|/(n + 2)$
0	0.866025	11	0.631936	22	0.650463
1	0.645497	12	0.661039	23	0.633685
2	0.735335	13	0.632330	24	0.649358
3	0.632745	14	0.657808	25	0.633865
4	0.698463	15	0.632679	26	0.648416
5	0.631095	16	0.655329	27	0.634025
6	0.681442	17	0.632983	28	0.647603
7	0.631135	18	0.653368	29	0.634168
8	0.671707	19	0.633249	30	0.646895
9	0.631513	20	0.651778	31	0.634297
10	0.665424	21	0.633481	32	0.646273

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