

ON A SCHUR-TYPE INEQUALITY IN L^2

P. GOETGHELUCK
 (Orsay)

Abstract. This note is a first attempt to give an admissible numerical value for the constant C in the estimate $\|P\|_2 \leq C n \|xP\|_2$ where P is an algebraic polynomial of degree at most n ($n > 0$) and $\|\cdot\|_2$ denotes the L^2 -metric on $[-1, 1]$.

1. Notation and result

Let f be a measurable function defined on $[-1, 1]$.

$$\text{We set } \|f\|_2 = \left[\int_{-1}^1 |f(x)|^2 dx \right]^{1/2},$$

$$\|f\|_\infty = \text{ess sup}_{x \in [-1, 1]} |f(x)|.$$

We denote by \mathbb{P}_n the set of algebraic polynomials of degree at most n . In 1919, Schur [4] gave the following estimate

$$\text{for any } P \in \mathbb{P}_n, \|P\|_\infty \leq (n+1) \|xP\|_\infty.$$

It is also well known (see for example [2] or [3 §6.3]) that there exists an absolute constant C such that for any $P \in \mathbb{P}_n$ ($n > 0$)

$$\|P\|_2 \leq C n \|xP\|_2. \quad (1)$$

Furthermore exponents 1 of n in (1) is optimal.

(More generally, analogous estimates hold in the L^2 -metric). This type of inequality is extensively used in approximation theory. Unfortunately a numerical value for the constant C has never been provided. So our aim is to prove the following.

Proposition. For any $P \in \mathbb{P}_n$ ($n \geq 0$)

$$\|P\|_2 \leq 1.13 (n+2) \|xP\|_2.$$

2. Proof of the proposition.

We recall that the classical Legendre polynomials (P_i) satisfy

$$iP_i = (2i-1)xP_{i-1} - (i-1)P_{i-2} \quad (i \geq 2)$$

$$\|P_i\|_\infty = (i + (1/2))^{-1/2} \quad (i \geq 0)$$

Then if we denote by (p_i) the associated set of orthonormal Legendre polynomials we have

$$p_i = \left(4 - \frac{1}{i^2}\right)^{1/2} x p_{i-1} - \left(1 + \frac{1}{2i^3 - 3i^2}\right)^{1/2} p_{i-2} \quad (i \geq 2). \quad (2)$$

Let us define a linear mapping T_n from \mathbb{P}_n into \mathbb{P}_{n-1} ($n > 0$) by

$$T_n: \alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_1 x + \alpha_0 \rightarrow \alpha_n x^{n-1} + \dots + \alpha_2 x + \alpha_1$$

that is $T_n: P \rightarrow Q$, where Q is the quotient in the euclidian division of P by x . Using (2) repeatedly shows that entries a_{ij} of the matrix $A = (a_{ij})_{0 \leq i < n, 0 \leq j < n}$ of the operator T_n with respect to the orthonormal basis (p_i) can be computed by the following formulas using auxiliary constants b_{ij}

$$\begin{aligned} b_{ij} &= 0 \text{ if } i > j \text{ or } i + j \text{ odd,} \\ b_{jj} &= 1 \quad (j = 1, 2, \dots, n) \\ b_{i-2,j} &= -b_{ij} \left(1 + \frac{1}{2i^3 - 3i^2}\right)^{1/2} \end{aligned} \quad (3)$$

$$a_{ij} = b_{ij} \left(4 - \frac{1}{i^2}\right)^{1/2} \quad (4)$$

$$(i = 1, \dots, n; j = 0, \dots, n)$$

Example: For $n = 3$, $A = \begin{pmatrix} 0 & \sqrt{3} & 0 & -2\sqrt{7}/3 \\ 0 & 0 & \sqrt{15}/2 & 0 \\ 0 & 0 & 0 & \sqrt{35}/3 \end{pmatrix}$.

We recall (see [1]) that

$$\|T_n\| = \text{Max} \{ \|T_n P\|_2 / \|P\|_2; P \in \mathbb{P}_n, P \neq 0 \}$$

is the square root of the largest eigenvalue of the matrix $'A.A$ or $'B.B$ where $B = (a_{ij})_{0 \leq i < n, 1 \leq j < n}$ (since the first column of A contains only 0-entries).

Example. With previous matrix A ($n = 3$) we get

$$\|T_1\| = \sqrt{3}, \quad \|T_2\| = \sqrt{15}/2, \quad \|T_3\| = (5 + \sqrt{40/3})^{1/2}.$$

In order to give an estimation of the largest eigenvalue of $'B.B$ we need an evaluation of entries a_{ij} of B . Using inequality (3) we get for any i and j

$$|b_{ij}| \leq \prod_{p=1}^{\infty} \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2} \quad (j \text{ even})$$

$$|b_{ij}| \leq \prod_{p=1}^{\infty} \left(1 + \frac{1}{(2p+1)^2(4p-1)}\right)^{1/2}$$

$$\leq \prod_{p=1}^{\infty} \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2} \quad (j \text{ odd}).$$

We have

$$\prod_{p=N+1}^{\infty} \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2} \leq \prod_{p=N+1}^{\infty} \left(1 + \frac{1}{8p^2(4p-3)}\right)$$

$$= \exp\left(\sum_{N+1}^{\infty} \text{Ln}\left(1 + \frac{1}{8p^2(4p-3)}\right)\right) \leq \exp\left(\sum_{N+1}^{\infty} \frac{1}{8p^2(4p-3)}\right)$$

$$\leq \exp\left(\int_N^{\infty} \frac{dx}{8x^2(4x-3)}\right) = \left(\frac{4N}{4N-3}\right)^{1/18} \exp(-1/(24N)).$$

Then for any i, j and N we have

$$|b_{ij}| \leq \left(\frac{4N}{4N-3}\right)^{1/18} \exp(-1/(24N)) \prod_{p=1}^N \left(1 + \frac{1}{4p^2(4p-3)}\right)^{1/2}$$

Taking $N = 10$ yields $|b_{ij}| < 1.13$. It follows from (4) that for any i and j we have $|a_{ij}| < 2.26$. So the matrix B has the following form

		x	0	x	0	x	\dots	
		0	x	0	x	0	\dots	
		0	0	x	0	x	\dots	
		0	0	0	x	0		
		0	0	0	0	0	x	

where x denotes a non zero term whose absolute value is less than 2.26.

For $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ we set $\|v\| = (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2}$ and $Bv = ((Bv)_1, (Bv)_2, \dots, (Bv)_n)$. We have

$$|(Bv)_1| \leq 2.26 \|(1, 0, 1, 0, \dots)\| \|v\|$$

$$|(Bv)_2| \leq 2.26 \|(0, 1, 0, 1, \dots)\| \|v\|$$

$$|(Bv)_3| \leq 2.26 \|(0, 0, 1, 0, \dots)\| \|v\|$$

and so on.

If n is even then

$$\|Bv\| \leq 2.26 (1 + 1 + 2 + 2 + \dots + (n/2) + (n/2))^{1/2} \|v\|$$

$$= 2.26 ((n/2) ((n/2) + 1))^{1/2} \|v\| \leq 1.13 (n + 1) \|v\|$$

and if n is odd

$$\|Bv\| \leq 2.26 (1 + 1 + 2 + 2 + \dots + ((n-1)/2) + ((n-1)/2) + ((n+1)/2)^{1/2} \|v\| = 1.13 (n + 1) \|v\|$$

so in both cases for any $v \in \mathbb{R}^n$, $\|Bv\| \leq 1.13 (n + 1) \|v\|$ and by a similar calculation $\|{}^t Bv\| \leq 1.13 (n + 1) \|v\|$ whence

$$\|{}^t B.Bv\| \leq 1.13^2 (n + 1)^2 \|v\|.$$

The last estimate shows that the largest eigenvalue of ${}^t B.B$ is less than $1.13^2 (n + 1)^2$ and then $\|T_n\| \leq 1.13 (n + 1)$. In particular for any polynomial $xP(x)$ with $P \in \mathbb{P}_n (n \geq 0)$ we have

$$\|P\|_2 = \|T_{n+1}(xP)\|_2 \leq \|T_{n+1}\| \|xP\|_2 \leq 1.13 (n + 2) \|xP\|_2$$

which completes the proof.

In Table 1 we give numerical values of $\|T_{n+1}\|/(n + 2)$ for n in the range $[0, 32]$. The calculation was made by Dörfler's method [1] (computation of the square root of the largest eigenvalue of ${}^t A.A$) with a computer. This table shows that the coefficient 1.13 is not optimal!

Table 1

n	$\ T_{n+1}\ /(n + 2)$	n	$\ T_{n+1}\ /(n + 2)$	n	$\ T_{n+1}\ /(n + 2)$
0	0.866025	11	0.631936	22	0.650463
1	0.645497	12	0.661039	23	0.633685
2	0.735335	13	0.632330	24	0.649358
3	0.632745	14	0.657808	25	0.633865
4	0.698463	15	0.632679	26	0.648416
5	0.631095	16	0.655329	27	0.634025
6	0.681442	17	0.632983	28	0.647603
7	0.631135	18	0.653368	29	0.634168
8	0.671707	19	0.633249	30	0.646895
9	0.631513	20	0.651778	31	0.634297
10	0.665424	21	0.633481	32	0.646273

REFERENCES

1. P. Dörfler, *New inequalities of Markov type*, SIAM J. Math. Anal. **18** (1987), 490–494.
2. P. Goetgheluck, *Polynomial inequalities and Markov's inequality in weighted L^p -spaces*, Acta Math. Acad. Sci. Hungar. **33** (1979), 325–331.
3. P. Nevai, "Orthogonal polynomials," Memoirs of the AMS 213 Providence RI USA, 1979.
4. I. Schur, *Über das maximum des absoluten Betrages eines Polynoms in einen gegebenen Intervall*, Math. Z. **4** (1919), 271–287.

Received 25.XII.1990

Université de Paris-sud,
Mathématiques Bât. 425,
91405 Orsay-Cedex
France

0 Introduction. Beginning with the fundamental work of V. Volterra [9] and A. Lotka [8], mathematical models of population dynamics, as well as competition, have been developed and generalized in various forms: the populations have been considered structured on age (see [1, 2, 3]), beside their natural growth, their diffusion in the space has been taken into account (see [4, 5, 6, 7]), their density was considered in the framework of the averaging theory [10], etc.

The present paper is concerned with a mathematical model of competition between two populations P_1 and P_2 , structured on ages, diffusing on \mathbb{R} , and starting initially from two disjoint domains, namely R_1 and R_2 . Moreover, the two populations are supposed to have different "activities", i.e. different diffusion coefficients, different death rates, and different fertility functions. As a consequence, the behaviour of the system will depend on these parameters. In addition, it is supposed that the death rate of the second population P_2 is increased by the total first population existing in the environment. In a special case, such a model could approximate the progress of a desert vegetation area, in a green land.

1. The model, as said, consists on two populations P_1, P_2 , each of them structured on ages, so that:

1. The population P_1 , with density u , diffuses on \mathbb{R} , has a constant diffusion coefficient d_1 and a death coefficient μ_1 . We suppose that the evolution of P_1 is not influenced by P_2 .

If we denote by t the time, then its diffusion equation is

$$(1.1) \quad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = d_1 \frac{\partial^2 u}{\partial x^2} - \mu_1 u = 0,$$

with the initial distribution on ages

$$(1.2) \quad u_1(x, 0, a) = \begin{cases} u_1(x, a) & \text{if } x \in R_1 \\ 0 & \text{if } x \in R_2 \end{cases}$$

2. The population P_2 diffuses also on \mathbb{R} , has h_1 as density, constant diffusion coefficient d_2 , death coefficient μ_2 but this last coefficient