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A MATHEMATICAL MODEL OF COMPETITION BETWEEN TWO POPULATIONS, WITH DISJOINT STARTING DOMAINS

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0. Introduction. Beginning with the fundamental work of V. Volterra [9] and A. Lotka [8], mathematical models of population dynamics, as well as competition, have been developed and generalized in various forms: the populations have been considered structured on ages (see f. i. [3]), beside their natural growth, their diffusion in the space has been taken into account (see. f. i. [1], [4], [5]), their theory was considered in the framework of the semigroup theory [10], etc.

The present paper is concerned with a mathematical model of competition between two populations $\mathbf{P}_1$ and $\mathbf{P}_2$, structured on ages, diffusing on $\mathbf{R}$, and starting initially from two disjoint domains, namely R_- and R_+. Moreover, the two populations are supposed to have different "activities", i.e. different diffusion coefficients, different death rates, and different fertility functions. As a consequence, the behaviour of the system will depend on these parameters. In addition, it is supposed that the death rate of the second population $\mathbf{P}_2$ is increased by the total first population existing in the environment. In a special case, such a model could approximate the progress of a desert vegetation area, on a green land.

1. The model, as said, consists on two populations $\mathbb{P}_1$, $\mathbb{P}_2$, each of them structured on ages, so that:

1. The population $\mathbf{P}_1$, with density u_1, diffuses on R, has a constant diffusion coefficient A_1^2 and a death coefficient λ_1. We suppose that the evolution of $\mathbf{P}_1$ is not influenced by $\mathbf{P}_2$.

If we denote by t the time, then its diffusion equation is

$$\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial a}-A_1^2\frac{\partial^2 u_1}{\partial x^2}+\lambda_1 u_1=0, \tag{1.1}$$

with the initial distribution on ages:

$$u_1(x, 0, a)=\begin{cases}\varphi_1(x, a) & \text{for } x\leq 0,\\ 0 & \text{for } x>0.\end{cases} \tag{1.2}$$

2. The population $\mathbf{P}_2$ diffuses also on R, has u_2 as density, constant diffusion coefficient A_2^2, death coefficient λ_2 — but this last coefficient

multiplied by $U_1(x, t)$ — the totality of the population u_1, namely

$$(1.3)\qquad U_1 = \int_0^\infty u_1(x, t, a)\, da.$$

The diffusion equation of P_2, will be :

$$(1.4)\qquad \frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial a} - A_2^2 \frac{\partial^2 u_2}{\partial x^2} + \lambda_2 U_1(x, t)\, u_2 = 0,$$

with initial density

$$(1.5)\qquad u_2(x, 0, a) = \begin{cases} \varphi_2(x, a) & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0. \end{cases}$$

3. Beside these equations, as usually, we must consider the two offspring equations :

$$(1.6)\qquad u_1(x, t, 0) = \psi_1(x,t) = \int_0^\infty \mu_1(a) u_1(x, t, a)\, da,$$

$$(1.7)\qquad u_2(x, t, 0) = \psi_2(x,t) = \int_0^\infty \mu_2(a)\, u_2(x, t, a)\, da,$$

where $\mu_i(a)$ are the fertilities of the two populations, supposed depending on their ages.

For $t = 0$, these relations lead to the compatibility equations :

$$(1.8)\qquad \varphi_1(x, 0) = \int_0^\infty \mu_1(a)\, \varphi_1(x, a)\, da,$$

$$(1.9)\qquad \varphi_2(x,0) = \int_0^\infty \mu_2(a)\, \varphi_2(x, a)\, da.$$

2. **Hypotheses.** 1. As said, the diffusion coefficients A_i^2 are constants ;

2. The death coefficients λ_i are constants.

3. The fertility functions are integrable functions on R_+.

4. Moreover, we will suppose that the fertility functions have the form

$$\mu_i(a) = h_i a e^{-k_i a},\ h_i k_i \text{, positive constants.}$$

5. The initial functions φ_i are integrable on their domain of definition.

For all classical notions and propositions regarding parabolic equations see [7] and [2].

3. **A. The system (1.1), (1.2), (1.6).** It is easily seen that the solution of (1.1), (1.2) for $a \geqslant t$, is :

$$(3.1)\qquad u_1(x, t, a) = e^{-\lambda_1 t} \int_{-\infty}^{0} E_1(x, t, y, 0)\, \varphi_1(y, a - t) dy$$

and for $a < t$:

$$(3.2)\qquad u_1(x, t, a) = e^{-\lambda_1 a} \int_{-\infty}^{0} E_1(x, a, y, 0)\, \psi_1(y, t - a) dy,$$

where $\psi_1(x,t)$ is, for the moment the unknown function from (1.6), supposed positive, integrable for each t on R_+ of class C^1 for each x, and

$$(3.3)\qquad E_i(x, t; y, s) = \frac{1}{2A_i\sqrt{\pi(t - s)}} \exp\left(-\frac{(x - y)^2}{4A_i^2(t - s)}\right).$$

Replacing (3.1), (3.2) in (1.6), we obtain the integral equation

$$(3.4)\qquad \psi_1(x,t) = \int_0^t ds \int_{-\infty}^{\infty} K_1(x, t; y, s)\, \psi_1(y, s)\, dy + F_1(x, t),$$

where

$$K_1(x, t; y, s) = \bar{\mu}_1(t - s)\, E_1(x, t, y, s),$$

$$(3.5)\qquad F_1(x,t) = \int_0^\infty \bar{\mu}_1(s + t) e^{\lambda_1 s} ds \int_{-\infty}^{0} E_1(x, t; y, 0)\, \varphi_1(y, s)\, dy$$

$$\bar{\mu}_1(a) = \mu_1(a) e^{-\lambda_1 a} = h_1 a\, e^{-(k_1+\lambda_1)a}.$$

It is now easy to check that the resolvent kernel of (3.4), is :

$$(3.6)\qquad \mathbf{K}_1(x, t; y, s) = M_1(t-s) E_1(x, t; y, s),$$

with

$$M_1(t-s) = \sum_{n=1}^{\infty} \bar{\mu}_1^{(n)}(t-s),\quad \bar{\mu}_1^{(n)}(t-s) = \int_s^t \bar{\mu}_1(t-b)\, \bar{\mu}_1^{(n-1)}(b-s)\, db,\quad \bar{\mu}_1^{(1)} = \bar{\mu}_1,$$

so that

$$M_1(t-s) = \sqrt{h_1} \sum_1^{\infty} \frac{(\sqrt{h_1}(t-s))^{2n-1}}{(2n-1)!} e^{-(\lambda_1+k_1)(t-s)},$$

and after some direct computations :

$$\psi_1(x,t) = \int_0^{\infty} Q_1(t,s) e^{\lambda_1 s} \, ds \int_{-\infty}^{0} E_1(x,t,y,0) \, \varphi_1(y,s) \, dy, \tag{3.7}$$

where

$$Q_1(t,s) = \frac{\sqrt{h_1}}{2} e^{-(\lambda_1+k_1)(t+s)} [(\sqrt{h_1}s+1) \, e^{\sqrt{h_1}t} + (\sqrt{h_1}s-1) \, e^{-\sqrt{h_1}t}], \tag{3.8}$$

and so, (for $a < t$)

$$u_1(x,t,a) = e^{-\lambda_1 a} \int_0^{\infty} Q_1(t-a,s) e^{\lambda_1 s} \, ds \int_{-\infty}^{0} E_1(x,t;y,0) \, \varphi_1(y,s) dy. \tag{3.9}$$

From (3.1) and (3.7), we obtain :

$$U_1(x,t) = \int_0^{\infty} R_1(t,s) \, ds \int_{-\infty}^{0} E_1(x,t;y,0) \, \varphi_1(y,s) \, dy, \tag{3.10}$$

with

$$R_1(t,s) = \frac{\sqrt{h_1}}{2} \left\{ \frac{\sqrt{h_1}s+1}{\sqrt{h_1}+k_1} e^{-k_1 s} e^{p_1 t} - \frac{\sqrt{h_1}s-1}{\sqrt{h_1}+k_1} e^{-k_1 s} e^{p_2 t} + \right. \tag{3.11}$$

$$\left. + \left(\frac{2}{\sqrt{h_1}} - \frac{2h_1(k_1 s+1)}{h_1-k_1^2} e^{-k_1 s} \right) e^{-\lambda_1 t} \right\},$$

$$p_1 = \sqrt{h_1} - \lambda_1 - k_1, \quad p_2 = -\sqrt{h_1} - \lambda_1 - k_1.$$

Supposing moreover

$$\varphi_1(x,a) = \gamma_1(x) \, \nu_1(a), \tag{3.12}$$

$U_1(x,t)$ takes the form :

$$U_1(x,t) = (g_1 e^{p_1 t} + g_2 e^{p_2 t} + g_3 e^{-\lambda_1 t}) \int_{-\infty}^{0} E_1(x,t;y,0) \, \gamma_1(y) \, dy, \tag{3.13}$$

and

$$g_1 = \frac{\sqrt{h_1}}{2} \int_0^{\infty} e^{-h_1 s} \frac{\sqrt{h_1}s+1}{\sqrt{h_1}-k_1} \nu_1(s) \, ds \, ; \; g_2 = -\frac{\sqrt{h_1}}{2} \int_0^{\infty} e^{-k_1 s} \frac{\sqrt{h_1}s-1}{\sqrt{h_1}+k_1} \nu_1(s) \, ds,$$

(3.14)

$$g_3 = \frac{\sqrt{h_1}}{2} \int_0^{\infty} \left(\frac{2}{\sqrt{h_1}} + \frac{2h_1(k_1 s+1)}{k_1^2-h_1} e^{-k_1 s} \right) \nu_1(s) ds.$$

As a conclusion of this paragraph :

The solution of system (1.1), (1.2), (1.6) is given by (3.1), (3.9), (3.7) and the total population by (3.10) or, with the hypothesis (3.12), by (3.13).

Remark. For $a \geq t$ the solution does not imply the function ψ_1, (i.e. the fertility), since the population with age $a \geq t$ is already born at $t = 0$, so that it has no relation with the fertility coefficient.

3. B. Existence of the solution of the system (1.3), (1.4), (1.7)

a) In the following, we will assume the hypothesis (3.12). First, for $a > t$, we perform the substitution

$$a - t = \alpha, \quad t = \tau,$$

and denote

$$u_2(x,t,a) = u_2(x,\tau,\alpha+\tau) = v_2(x,\tau,\alpha) = v_2(x,t,a-t).$$

Equations (1.3), (1.4) become :

$$\frac{\partial v_2}{\partial \tau} - A_2^2 \frac{\partial^2 v_2}{\partial x^2} + \lambda_2 \, U_1(x,\tau) \, v_2 = 0,$$

$$v_2(x,0,\alpha) = \begin{cases} \varphi_2(x,\alpha) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Since the fundamental solution of this system implies an alternate series inadequate for approximation, we take

$$v_2 = V_2 \exp(-c \, e^{p_1 \tau}) \tag{3.15}$$

(c a new constant, p_1 given by (3.11)). V_2 satisfies the equations :

$$\frac{\partial V_2}{\partial \tau} = A_2^2 \frac{\partial^2 V_2}{\partial x^2} + (\lambda_2 U_1(x,\tau) - c p_1 \, e^{p_1 \tau}) \, V_2 = 0,$$

(3.16)

$$V_2(x,0,\alpha) = \begin{cases} \varphi_2(x,\alpha) \, e^{-c} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Supposing $p_1 \neq 0$, we choose now $c = \lambda_2 g_1/p_1$, so that

$$cp_1\, e^{p_1\tau} - \lambda_2 U_1(x,\ \tau) = H_0(x,\ \tau) \leqslant r\ \lambda_1\, e^{-\lambda_1\tau},$$

r — a constant.

The solution of equations (3.16) is :

$$V_2(x,\ \tau,\ \alpha) = \int_0^\infty \mathbf{R}_1(\alpha\, ;\ x,\ \tau\, ;\ y,\ 0)\ \ \varphi_2(y,\ \alpha) e^c\ dy,$$

R_1 being the fundamental solution of the operator defined by the first member of equation (3.16). This solution can be approximated from the equivalent with (3.16) integral equation :

$$V_2(x,\ \tau,\ \alpha) = \int_0^\infty E_2(x,\ \tau\, ;\ y,\ 0)\ \varphi_2\ (y,\ \alpha) e^c\ dy +$$

$$+ \int_0^\tau d\sigma \int_{-\infty}^\infty E_2(x,\ \tau\, ;\ y,\ \sigma)\ H_0(y,\ \sigma)\ V_2(y,\ \sigma,\ \alpha)\ dy.$$

We obtain so :

$$V_2(x,\ \tau,\ \alpha)\ \exp(c + r)\ \exp(-\, re^{-\lambda_1\tau}) \int_0^\infty E_2(x,\ \tau\, ;\ y,\ 0)\ \varphi_2\ (y,\ \alpha)\ dy,$$

and, then :

$$v_2(x,\ \tau,\ \alpha) \leqslant \Lambda_1(\tau) \int_0^\infty E_2(x,\ \tau\, ;\ y,\ 0) \varphi_2(y,\ \alpha)\ dy,$$

where

$$\Lambda_1(\tau) = \exp\{-r(e^{-\lambda_1\tau} - 1) - (\lambda_2\, g_1/p_1)\ (e^{p_1\tau} - 1)\}\, ;$$

finally

(3.17) $$u_2(x,\ t,\ a) \leqslant \Lambda_1(t) \int_0^\infty E_2(x,\ t\, ;\ y,\ 0) \varphi_2(y,\ a - t)\ dy.$$

b) For $a < t$, we perform the substitution

$$t - a = \tau, \quad a = \alpha$$

and denote

$$u_2(x,\ t,\ a) = u_2(x,\ \alpha + \tau,\ \alpha) = w_2(x,\ \tau,\ \alpha) = w_2(x,\ t - a,\ a).$$

(1.3), (1.4) become :

$$\frac{\partial w_2}{\partial \alpha} - A_2^2 \frac{\partial^2 w_2}{\partial x^2} + \lambda_2 U_1(x, \alpha + \tau) w_2 = 0,$$

$$w_2(x,\ \tau,\ 0) = \psi_2(x,\ \tau).$$

As in paragraph a), we take

$$w_2(x,\ \tau,\ \alpha) = W_2(x, \tau, \alpha)\ \exp(-d e^{p_1(\alpha+\tau)}),\ d - \text{a constant},$$

The function W_2 satisfies

(3. 18) $$\frac{\partial W_2}{\partial \alpha} - A_2^2 \frac{\partial^2 W_2}{\partial x^2} + (\lambda_2 U_1(x,\ \alpha + \tau) - dp_1\, e^{p_1(\alpha+\tau)})\ W_2 = 0,$$

$$W_2(x,\ \tau,\ 0) = \psi_2(x,\ \tau) e^d$$

Take now $d = \lambda_2 g_1/p_1$, to obtain

$$dp_1\, e^{p_1(\alpha+\tau)} - \lambda_2 U_1(x,\ \alpha + \tau) \leqslant s\lambda_2\, e^{-\lambda_2(\alpha+\tau)}.$$

(s — a new constant).

Using the fundamental solution of the operator defined by (3.18), the solution of this system is :

$$W_2(x,\ \tau,\ \alpha) = \int_{-\infty}^\infty \mathbf{R}_2(\tau\, ;\ x,\ \alpha\, ;\ y, 0)\ \psi_2(x,\ \tau)\ e^d\ dy,$$

where, as in paragraph a), R_2 can be approximated by the intermediate of an integral equations ; we get so :

$$W_2(x,\ \tau,\ \alpha) \leqslant \Lambda_2(\alpha + \tau) \int_{-\infty}^\infty E_2(x,\ \alpha,\ y,\ 0)\ \ \psi_2(y,\ \tau)\ dy,$$

with

$$\Lambda_2(\tau) = \exp\{-\, s(e^{-\lambda_1\tau} - 1) - (\lambda_2\, g_1/p_1)(e^{p_1\tau} - 1)\}$$

and finally :

(3.19) $$u_2(x,t,a) \leqslant \Lambda_2(t) \int_{-\infty}^\infty E_2(x,\ a\, ;\ y,\ 0)\ \psi_2(y,\ t - a)\ dy.$$

c) We come now to equation (1.7), where, using the above estimates for $\mathbf{R}_1$ and $\mathbf{R}_2$, we obtain :

$$\psi_2(x,t) \leqslant \int_0^t \mu_2(a)\mathrm{d}a \int_0^\infty \Lambda_2(t)\ E_2(x, a;\ y,0)\ \psi_2(y, t-a)\mathrm{d}y +$$

$$+ \int_t^\infty \mu_2(a)\mathrm{d}a \int_0^\infty \Lambda_1(t)\ E_2(x,\ t;\ y,\ 0)\ \varphi_2(y,\ a-t)\mathrm{d}y,$$

or

$$\psi_2(x,t) \leqslant \Lambda_2(t)\int_0^\infty h_2(t-a)\mathrm{e}^{-k_2(t-a)}\ \mathrm{d}a\int_{-\infty}^\infty E_2(x, t-a;\ y,0)\ \psi_2(y,\ a)\ \mathrm{d}y +$$

$$+ \Lambda_1(t)\int_0^\infty h_2(a+t)\ \mathrm{e}^{-k_2(a+t)}\ \mathrm{d}a\int_0^\infty E_2(x,t;\ y,\ 0)\ \varphi_2(y,\ a)\ \mathrm{d}y,$$

which leads to

$$\text{(3.19)}\qquad \psi_2(x,\ t) \leqslant \frac{\sqrt{h_1}}{2}\ \Lambda_2(t)\ \mathrm{e}^{-k_2 t}\int_0^\infty \mathrm{e}^{-k_2 s}\ \{(\sqrt{h_2}s+1)\ \mathrm{e}^{\sqrt{h_2}t} +$$

$$+ (\sqrt{h_2}s-1)\ \mathrm{e}^{-\sqrt{h_2}t}\ \}\ \mathrm{d}s\int_0^\infty E_2(x,\ t;\ y,0)\ \varphi_2(y,\ s)\ \mathrm{d}y,$$

and finally, (for $a < t$)

$$\text{(3.20)}\qquad u_2(x,\ t,a) \leqslant \frac{\sqrt{h_2}}{2}\ \Lambda_2(t-a)\ \mathrm{e}^{-k_2(t-a)}\int_0^\infty \mathrm{e}^{-k_2 s}\ \{(\sqrt{h_2}s+1)\ \mathrm{e}^{\sqrt{h_2}(t-a)} +$$

$$+ (\sqrt{h_2}s-1)\ \mathrm{e}^{-\sqrt{h_2}(t-a)}\}\ \mathrm{d}s.\int_0^\infty E_2(x,\ t;\ y,\ 0)\ \varphi_2(y,s)\ \mathrm{d}y.$$

As a conclusion :

The solution u_2 of the system (1.3), (1.4), (1.7), is estimated for $a \geqslant t$ by the expression (3.15) and for $a < t$ by (3.20) and the function ψ_2, by (3.19).

4. Conclusions concerning the behaviour of the two populations.

From the estimates of u_2 for $a < t$ and $a \geqslant t$, we deduce an estimate of the total second population U_2; an estimate of the first population is given by (3.13) always with the hypothesis (3.12). We get, obviously

$$\text{(4.1)}\qquad U_2(x,t) \leqslant \frac{\sqrt{h_2}}{2}\int_0^t \Lambda_2(t-a)\mathrm{e}^{(\sqrt{h_2}-k_2)(t-a)}\ \mathrm{d}a.$$

$$\int_0^\infty \mathrm{e}^{-k_2 s}\ \{(\sqrt{h_2}\, s+1) + (\sqrt{h_2}s-1)\ \mathrm{e}^{-2\sqrt{h_2}(t-a)}\}\ \mathrm{d}s\int_0^\infty E_2(x,\ t;\ y,0)\ \varphi_2(y,\ s)\ \mathrm{d}y$$

$$+ \Lambda_1(t)\int_0^\infty \mathrm{d}a\int_0^\infty E_2(x,\ t;\ y,a)\varphi_2\ (y,a)\ \mathrm{d}y.$$

Suppose now, first $p_1 > 0$. It follows from (3.13) that $\mathbf{P}_1$, tends to infinity as $t \to \infty$. Then, taking into account the expressions of Λ_1 and Λ_2, it is easily seen that

$$\text{(4.2)}\quad U_2(x,t) \leqslant K\exp[-r\,(\mathrm{e}^{-\lambda_1 t}-1) - (\lambda_2\, g_1/p_1)\ (\mathrm{e}^{p_1 t}-1) + (\sqrt{h_2}-h_2)t]$$

(K — a constant) that tends to zero as $t \to \infty$; in other words, the second population is overgrown by the first one.

If $p_1 < 0$, the first population tends to zero as $t \to \infty$; the second, as seen from (4.2), tends to infinity when $\sqrt{h_2}-k_2 > 0$, i.e. when the fertility rate of $\mathbf{P}_2$ is sufficiently large, but remains bounded, eventualy tends to zero, as $\sqrt{h_2}-k_2 < 0$.

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