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A SUFFICIENT UNIVALENCE CRITERION FOR HOLOMORPHIC FUNCTIONS

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Let $f(z)$ be a function holomorphic in $|z|<1$.
One supposes without any loss of generality that $f(0) = 0$ and $f'(0) = 1$.

Let now consider the Taylor development of $f(z)$, development which is absolutely and uniformly convergent on any compact subsets contained in $|z| < 1$, i.e.

$$f(z) = \sum_{\gamma=1}^{\infty} \frac{f^{(\gamma)}(0)}{\gamma\,!} z^{\gamma} \equiv \sum_{\gamma=1}^{\infty} \alpha_{\gamma} z^{\gamma}, \text{ with } \alpha_1 \equiv f'(0) = 1.$$

Obviously the sequence of partial sums of this Taylor series $\{\alpha_{\gamma} z^{\gamma}\}_{\gamma \in N}$ is a sequence of polynomials which, under the hypothesis of their univalence and taking also into account the locally uniformly convergence of the sequence, leads to the univalence in $|z| < 1$ of the sum function $f(z)$.

On the other hand J. Dieudonné gave in 1931 [1] a necessary and sufficient univalence criterion in $|z|<1$ for polynomials. Precisely according to this criterion a polynomial $\sum_{\gamma=0}^{n} \alpha_{\gamma} z^{\gamma}$ is univalent iff their coefficient satisfies the condition:

$\sum_{\gamma=0}^{n} \frac{\sin \gamma\theta}{\sin \theta} z^{\gamma-1} = 0$, $(\forall) z \in \{z : |z| < 1\}$ and $(\forall)\theta \in \left[0, \frac{\pi}{2}\right]$. Following this result if the above-mentioned condition is fullfiled for $(\forall) n \in N$, the coefficients being now $\frac{f^{(\gamma)}(0)}{\gamma\,!}$, we would have a sufficient univalence condition in $|z| < 1$ for the given function $f(z)$.

At the same time we also remark that the condition

$$\sum_{\gamma=1}^{n} \frac{f^{(\gamma)}(0)}{\gamma\,!} \frac{\sin \gamma\theta}{\sin \theta} \cdot z^{\gamma-1} \neq 0 \text{ in } |z| < 1$$

should be implied by the fact that all the roots z_i of this polynomial belong to the outside of the unit disk $|z| = 1$, that means that $|z_i| > r \geq 1$, r being the lower limit of the distances of these roots to the origin. But it is known [2] that this r satisfies the following algebrical equation with real coefficients:

$P(r) \equiv |a_n| r^{n-1} + |a_{n-1}| r^{n-2} + \ldots + |a_2| r - |a_1| = 0$, where $a_{\gamma} = \frac{f^{(\gamma)}(0)}{\gamma\,!} \cdot \frac{\sin \gamma\theta}{\sin \theta}$

equation about what one can state that it has a unique positive real solution. Moreover this unique solution fullfils the condition $r \geqslant i$ iff $|a_n| + + |a_{n-1}| + \ldots + |a_2| \leqslant |a_1| = 1 \quad (\forall) n \in N$.

This result also assures the convergence of the series $\sum_{\gamma=2}^{\infty} |a_\gamma| \equiv \equiv \sum_{\gamma=2}^{\infty} \frac{f^{(\gamma)}(0)}{\gamma !} \frac{\sin \gamma\theta}{\sin \theta}$, remark important in the sequel.

We also led to the following theorem.

Theorem. *A sufficient univalence condition for the function $f(z)$ (i.e. $f(z) \in S$) is that for $(\forall) n \in N$ and $(\forall) \theta \in \left[0, \frac{\pi}{2}\right]$ the inequality $\sum_{\gamma=2}^{n} \cdot \left| \frac{f^{(\gamma)}(0)}{\gamma !} \cdot \frac{\sin \gamma\theta}{\sin \theta} \right| \leqslant 1$ holds i.e. the series sum $\sum_{\gamma=2}^{\infty} \left| \frac{f^{(\gamma)}(0)}{\gamma !} \cdot \frac{\sin \gamma\theta}{\sin \theta} \right|$ does not overpass the unity.*

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